Nonlinear filtering of one-dimensional diffusions in the case of a high signal-to-noise ratio

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ABSTRACT

In this paper, we are concerned with the asymptotic nonlinear filtering of one-dimensional diffusions as the observation noise tends to zero. The intensity of the signal noise may be normal, small or large. We derive evaluations of the conditional moments and obtain one- and two-dimensional approximate filters. We give upper bounds for the approximation errors and compare these filters with some classical suboptimal filters.

RESUME

Dans cet article, on etudie asymptotiquement le filtrage non linéaire des diffusions unidimensionnelles lorsque le bruit d'observation tend vers zero. L'intensité du bruit du signal peut être petite, grande ou de taille normale. On obtient des estimations sur les moments conditionnels ainsi que des filtres approchés de dimension 1 et 2. Des majorations sur les erreurs d'approximation de ces filtres et de quelques filtres classiques sont données, et les différents filtres sont comparés.

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1. Introduction

Consider the one-dimensional nonlinear filtering problem where the signal process X_t and the observation process Y_t satisfy

$$dX_t = b(X_t)dt + \varepsilon^{\gamma}\sigma(X_t)dW_t \tag{1.1}$$

$$dY_t = h(X_t)dt + \varepsilon^{1-\gamma}dB_t \tag{1.2}$$

In these equations, W_t and B_t are two mutually independent Brownian motions, ε and γ are real parameters, b, σ and h are real-valued functions (regularity assumptions will be precised subsequently). We will assume that h is one-to-one. First suppose that $\gamma=0$, so we have to study the problem

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \tag{1.3}$$

$$dY_t = h(X_t)dt + \varepsilon dB_t \tag{1.4}$$

If $\varepsilon=0$, then the trajectories of Y_t are differentiable and

$$X_t = h^{-1}(\dot{Y}_t) \tag{1.5}$$

So the signal process is exactly observed. Now, if ε is positive small but different from zero, the observations are slightly noisy and the filtering problem consists of computing, for Borel measurable functions g, the conditional mean $\hat{g}_t$ of $g(X_t)$ given the observations Y_s, $0 \leq s \leq t$; the solution of this problem is said to be finite-dimensional if it can be expressed by means of the solution of a finite-dimensional stochastic differential equation driven by the observation process Y. In the general case, the optimal filter is infinite-dimensional so we are interested in finding finite-dimensional approximations valid as $\varepsilon \to 0$. This is a singular perturbation problem: since the trajectories of Y_t are no more differentiable, one cannot use (1.5) as an approximate value for $\hat{X}_t$.

In [7], the authors consider the problem (1.3)-(1.4) with a constant function σ. Their method relies on Zakai's equation -a linear stochastic partial differential equation whose solution is an unnormalized version of the conditional density. They use some singular perturbation techniques on this equation to derive formally, as ε tends to 0, asymptotic expansions of the conditional density in the WKB form; then they deduce approximate finite-dimensional filters. As yet, these expansions have not been justified.

In this paper, we will prefer a method which does not involve Zakai's equation but two other basic tools of the nonlinear filtering theory, namely the Kallianpur-Striebel formula ([6]) and the semimartingale decomposition of $\widehat{g}_t$ for regular functions g ([9]). The Kallianpur-Striebel formula expresses $\widehat{g}_t$ as the quotient of two integrals over the space of signal trajectories; together with some Girsanov transformations and some results on the time reversal of diffusion processes, it will imply (theorem 5.1) that if M_t is the solution of

$$dM_t = b(M_t)dt + \frac{\sigma(M_t)}{\varepsilon}(dY_t - h(M_t)dt) \tag{1.6}$$

with some initial condition $M_0 = m_0 \in \mathbb{R}$, then

$$\widehat{X}_t = M_t + O(\varepsilon) \tag{1.7}$$

(the precise meaning of the term $O(\varepsilon)$ will be explained in section 2).

Another important application of this method will consist of getting estimates for further conditional moments of X_t. This problem has been met in [1]: defining

$$e_t \equiv X_t - \widehat{X}_t \tag{1.8}$$

asymptotic estimates on $\widehat{e_t^2}$ and $\widehat{e_t^3}$ were given in the case $\sigma(x)=\sigma_0$, $h(x)=x$ and under some stationarity assumptions. In this paper, we will generalize and improve these results. For instance, if σ and h' are uniformly positive and with some other regularity assumptions, it will be shown in section 6 that

$$\widehat{e_t^2} = \varepsilon \frac{\sigma}{h'}(\widehat{X}_t) + O(\varepsilon^{3/2}) \tag{1.9}$$

$$\widehat{e_t^3} = \varepsilon^2 \frac{2\sigma\sigma' h' - \sigma^2 h''}{h'^3}(\widehat{X}_t) + O(\varepsilon^{5/2}) \tag{1.10}$$

$$\widehat{e_t^4} = 3\varepsilon^2 \frac{\sigma^2}{h'^2}(\widehat{X}_t) + O(\varepsilon^{5/2}) \tag{1.11}$$

Then we will use these evaluations to obtain accurate filters; in [5], the semimartingale decomposition of $\widehat{g}_t$ was used for $g(x)=x$ and $g(x)=x^2$ to construct some second-order filters; our technique will be similar and, moreover, we will estimate the difference between approximate and optimal filters. For instance, it will be proved in theorem 7.1 that if (M_t, R_t) is the solution of

$$dM_t = b(M_t)dt + \frac{R_t}{\varepsilon}\sigma^2 h'(M_t)(dY_t - h(M_t)dt - \frac{\varepsilon}{2}\frac{\sigma h''}{h'}(M_t)dt) \tag{1.12}$$

$$\frac{dR_t}{dt} = -\frac{1}{\varepsilon}\sigma^2 h'^2(M_t)R_t^2 + 2(\frac{b}{\sigma})'\sigma(M_t)R_t + \frac{1}{\varepsilon} \tag{1.13}$$

with some initial condition $M_0=m_0$ and $R_0=r_0>0$, then

$$\widehat{X}_t = M_t + O(\varepsilon^{3/2}) \tag{1.14}$$

Some classical filters can also be studied with this approach: it will appear that the extended Kalman filter, the statistical linearization ([2]) and the modified second-order filter ([5]) yield an error of order ε. We will note that all these filters have short memory as ε tends to 0 (old values of the observation process are needless): this stability property is essential in the proofs; this will also imply that the error estimates will be uniform as t tends to infinity.

Now, what happens when $\gamma\neq 0$? In [4], the case $\gamma=1/2$ is considered; it is proved that the asymptotic problem is related to some control problem and a WKB expansion for the conditional density is derived. Here we will assume $\gamma<1/2$ because in this case, the filter has still a short memory and this condition is necessary for our method to work; the behaviour of the filter will be similar to the case $\gamma=0$. However, the accuracy of our evaluations will decrease as $\gamma\to 1/2$; for instance, (1.9)-(1.11) will hold for $\gamma\leq 1/4$. Note that we allow γ to be negative, and in this case, the signal process has a large covariance noise.

Let us outline the contents of the paper. In section 2, we list the assumptions and we derive some easy estimates. In section 3, we find a probabilistic representation of the a priori density of X_t; from this result and the Kallianpur-Striebel formula, we infer in section 4 a formula for the conditional density

$q(t,x)$ of X_t and we compute its derivative with respect to x. The one-dimensional filter (1.6), the estimates (1.9)-(1.11) and the two-dimensional filter (1.12)-(1.13) are respectively derived in sections 5, 6 and 7. In section 8, we study some classical filters and finally, we summarize the results in some tables.

2. Assumptions and first results

For every positive integer p, define $\underline{\underline{D}}(p)$ the space of real-valued functions φ defined on $\mathbb{R}$ such that φ is p times continuously differentiable and $\varphi^{(p)}(x)$ has polynomial growth as x tends to infinity. Define also $\underline{\underline{B}}(p)$ the subset of functions φ such that, for $1 \leq i \leq p$, $\varphi^{(i)}$ is bounded.

We will assume that the initial law of the signal process is absolutely continuous with respect to the Lebesgue measure, with density p_0. We will use the function a defined by $a = \sigma^2/2$. Now let us list the regularity assumptions on p_0 and on the coefficients b, σ and h. They are indexed by a positive integer n which will be specified for each result.

Assumption (An)

(i) *The functions b, σ, h and $\log(p_0)$ are respectively in $\underline{\underline{B}}(n+1)$, $\underline{\underline{B}}(n+2)$, $\underline{\underline{B}}(n+2)$ and $\underline{\underline{D}}(n)$; the functions b/σ and bh' are in $\underline{\underline{B}}(1)$.*

(ii) *The functions h' and σ are bounded and uniformly positive (i.e. they are bounded below by a strictly positive constant).*

(iii) *p_0 is a strictly positive bounded function such that for every real c,*

$$\int_{-\infty}^{+\infty} e^{cx} p_0(x)\,dx < \infty.$$

Note that the coefficients are not allowed to depend on the time. This more general case could also be studied with the method that we are going to describe, but the results would be slightly weaker and the formulas less readable... The minimal assumption will be (A0) and it will be assumed throughout this paper.

Now, let Ω^X and Ω^Y be two copies of the space of continuous functions from $[0,\infty[$ into $\mathbb{R}$, define $\Omega = \Omega^X \times \Omega^Y$ and (X_t, Y_t) the canonical process of Ω:

$$X_t(\omega_1,\omega_2) = \omega_1(t) \quad \text{and} \quad Y_t(\omega_1,\omega_2) = \omega_2(t)$$

If $0 \leq s \leq t$ and if Z_u is a Borel measurable process on Ω, then $\underline{\underline{F}}_t^s(Z)$ will denote the σ-algebra generated by Z_u, $s \leq u \leq t$, and we will note $\underline{\underline{F}}_t(Z) = \underline{\underline{F}}_t^0(Z)$, $\underline{\underline{F}}_t = \underline{\underline{F}}_t(X,Y)$ and $\underline{\underline{F}} = \underline{\underline{F}}_\infty$. Fix two real numbers $\varepsilon > 0$ and $\gamma < 1/2$. Assumption (A0) is amply sufficient to ensure the existence and the uniqueness of a solution P^X to the martingale problem defined on Ω^X by the generator

$$\underline{\underline{L}} = \varepsilon^{2\gamma} a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} \tag{2.1}$$

and the initial condition

$$P^X(X_0 \in dx) = p_0(x)\,dx \tag{2.2}$$

Let P^Y be the standard Wiener measure on Ω^Y and define on Ω the probabilities

$$\overset{\circ}{P} \equiv P^X \otimes P^Y \tag{2.3}$$

and P satisfying

$$\left.\frac{dP}{d\overset{\circ}{P}}\right|_{\underline{\underline{F}}_t} = \exp\left\{\frac{1}{\varepsilon^{2-2\gamma}}\left(\int_0^t h(X_s)\,dY_s - \frac{1}{2}\int_0^t h^2(X_s)\,ds\right)\right\} \tag{2.4}$$

for every $t \geq 0$. It follows from the Girsanov theorem that (X_t, Y_t) satisfies (1.1)-(1.2) where (W_t, B_t) is a $(\underline{\underline{F}}_t, P)$ Brownian motion. Remark that these probabilities depend on ε, so, since we are going to want ε to tend to 0, we should write a sub-script or superscript ε somewhere; nevertheless, for the readability of the equations, we will not do this. We will proceed similarly for processes depending on ε.

It will turn out that our problem involves a boundary layer at $t=0$; more precisely, we shall deal with processes Z_t which will be of order ε^k, but only on each time interval $[t_0, +\infty[$; for small values of t, they will be of order ε^r for some $r \leq k$ but the value of r will be needless for us. This is why we put the following definition: if Z_t is a measurable process (depending on ε) defined on Ω, we will write $Z_t = O(\varepsilon^k)$ if there exists $\varepsilon_0 > 0$ and r (which may be negative) such that for any $t_0 > 0$ and $1 \leq q < \infty$, there exists C such that.

$$\sup_{t_0 \le t < \infty} \|Z_t\|_q \le C\varepsilon^k \quad \text{and} \quad \sup_{0 \le t < \infty} \|Z_t\|_q \le C\varepsilon^r \tag{2.5}$$

for $\varepsilon<\varepsilon_0$ (note that ε_0 does not depend on q and t_0). With this definition, we can state and prove the following technical lemma.

Lemma 2.1 *Let Z_t be an $\underline{\underline{F}}_t$-adapted process such that for some r, each moment of Z_0 is dominated by ε^r, and, for some $\alpha>0$,*

$$dZ_t = \frac{1}{\varepsilon^\alpha}(-\zeta_t + A_t)dt + \frac{1}{\varepsilon^{\alpha/2}}d\rho_t \tag{2.6}$$

where

(*) *A_t is an $\underline{\underline{F}}_t$-adapted process satisfying*

$$A_t = O(\varepsilon^k) \tag{2.7}$$

(**) *ρ_t is an $\underline{\underline{F}}_t$ continuous martingale such that $<\rho,\rho>_t$ is absolutely continuous and*

$$\frac{d<\rho,\rho>_t}{dt} = O(\varepsilon^{2k}) \tag{2.8}$$

(***) *ζ_t is an $\underline{\underline{F}}_t$-adapted process such that, for some constant $C>0$,*

$$Z_t\zeta_t \ge C Z_t^2 \tag{2.9}$$

Then $Z_t = O(\varepsilon^k)$.

Proof

Fix an even integer $q>2$ and write Itô's formula

$$dZ_t^q = \frac{q}{\varepsilon^\alpha}\left(Z_t^{q-1}(-\zeta_t+A_t)+\frac{q-1}{2}Z_t^{q-2}\frac{d<\rho,\rho>_t}{dt}\right)dt + \frac{qZ_t^{q-1}}{\varepsilon^{\alpha/2}}d\rho_t \tag{2.10}$$

By means of classical martingale inequalities, one can deduce that the qth moment of Z_t is finite and

$$\frac{d}{dt}\mathbb{E}[Z_t^q] \le \frac{q}{\varepsilon^\alpha}\left(-C\,\mathbb{E}[Z_t^q]+\mathbb{E}[Z_t^{q-1}A_t]+\frac{q-1}{2}\mathbb{E}[Z_t^{q-2}\frac{d<\rho,\rho>_t}{dt}]\right) \tag{2.11}$$

On the other hand, if r and $\overline{r}$ are two positive numbers such that $1/r + 1/\overline{r} = 1$, by studying the function $(a,b)\to ab/(a^r+b^{\overline{r}})$ defined on $\mathbb{R}_+^2$, one can prove that this function is always bounded by 1. Applying this result with $r=q$, $a=(3/C)^{(q-1)/q}|A_t|$ and $b=(C/3)^{(q-1)/q}|Z_t|^{q-1}$, we obtain

$$|A_t Z_t^{q-1}| \leq \frac{C}{3} Z_t^q + (\frac{3}{C})^{q-1} A_t^q \tag{2.12}$$

Similarly, with $r=q/2$,

$$\frac{d<\rho,\rho>_t}{dt} Z_t^{q-2} \leq \frac{2C}{3(q-1)} Z_t^q + (\frac{3(q-1)}{2C})^{\frac{q}{2}-1} (\frac{d<\rho,\rho>_t}{dt})^{q/2} \tag{2.13}$$

Thus

$$\frac{d}{dt}\mathbb{E}[Z_t^q] \leq \frac{q}{\varepsilon^\alpha}\left[-\frac{C}{3}\mathbb{E}[Z_t^q] + (\frac{3}{C})^{q-1}\mathbb{E}[A_t^q] \right. \tag{2.14}$$

$$\left. + \frac{C}{3}(\frac{3(q-1)}{2C})^{q/2}\mathbb{E}[(\frac{d<\rho,\rho>_t}{dt})^{q/2}]\right]$$

From classical estimates on solutions of ordinary differential inequations, for any $t \geq s \geq 0$,

$$\mathbb{E}[Z_t^q] \leq \mathbb{E}[Z_s^q]\exp\{-\frac{Cq(t-s)}{3\varepsilon^\alpha}\} + \left[(\frac{3}{C})^q \sup_{s\leq u\leq t} \mathbb{E}[A_u^q] \right. \tag{2.15}$$

$$\left. + (\frac{3(q-1)}{2C})^{q/2} \sup_{s\leq u\leq t} \mathbb{E}[(\frac{d<\rho,\rho>_u}{du})^{q/2}]\right]$$

Now we use the assumptions on A and $d<\rho,\rho>/dt$ and we choose ε_0 sufficiently small so that it satisfies (2.5) for these two processes; the lemma then follows from (2.15) used first for $s=0$ to obtain the estimate on all the time interval and secondly for $s=t_0/2$ to obtain the estimate on $[t_0,+\infty[$. ■

The filtering problem consists of computing, under the probability P,

$$\hat{g}_t = \mathbb{E}[g(X_t)|\underline{\underline{F}}_t(Y)] \tag{2.16}$$

for some measurable functions g. We want to derive asymptotic expressions for this conditional mean, valid as ε tends to 0. Here is a family of filters with an error bound of order $\sqrt{\varepsilon}$:

Proposition 2.2 *Assume (A0). Let m_0 be a real number, let U_t and V_t be $\underline{\underline{F}}_t(Y)$-adapted processes (depending on ε) such that*

$$U_t = O(\varepsilon^{2\gamma-1/2}), \quad V_t = O(1) \tag{2.17}$$

and for some constant number $c>0$, $V_t \geq c$. *If* M_t *satisfies the equation*

$$dM_t = (b(M_t)+U_t)dt + \frac{V_t}{\varepsilon^{1-2\gamma}}(dY_t - h(M_t)dt) \tag{2.18}$$

with the initial condition $M_0 = m_0$, *then*

$$X_t - M_t = O(\sqrt{\varepsilon}) \tag{2.19}$$

Proof

The process $X-M$ is a solution of

$$d(X_t-M_t) = -\frac{V_t}{\varepsilon^{1-2\gamma}}(h(X_t)-h(M_t))dt + (b(X_t)-b(M_t))dt - U_t dt \tag{2.20}$$

$$+ \varepsilon^{\gamma}(\sigma(X_t)dW_t - V_t dB_t)$$

This equation can be written in the form (2.6) with $\alpha = 1-2\gamma$,

$$A_t = -\varepsilon^{1-2\gamma}U_t = O(\sqrt{\varepsilon}) \tag{2.21}$$

$$d\rho_t = \sqrt{\varepsilon}(\sigma(X_t)dW_t - V_t dB_t) \tag{2.22}$$

$$\zeta_t = V_t(h(X_t)-h(M_t)) - \varepsilon^{1-2\gamma}(b(X_t)-b(M_t)) \tag{2.23}$$

Therefore

$$(X_t-M_t)\zeta_t \geq (c\inf_x h'(x) - \varepsilon^{1-2\gamma}\sup_x b'(x))(X_t-M_t)^2 \tag{2.24}$$

The coefficient of the right side is positive if ε is sufficiently small so we can apply lemma 2.1 and prove the proposition. ■

The simplest choice for U and V is to take $U=0$ and $V = constant$, so an immediate corollary is:

Corollary 2.3 *Assume (A0). Let* m_0 *and* $\nu>0$ *be real numbers and define* M_t *the solution of the equation*

$$dM_t = b(M_t)dt + \frac{\nu}{\varepsilon^{1-2\gamma}}(dY_t - h(M_t)dt) \tag{2.25}$$

with the initial condition $M_0=m_0$. *Then* $X_t-M_t = O(\sqrt{\varepsilon})$.

Remarks

1. The initial value X_0-M_0 is not necessarily small. Actually, from inequation (2.15), there is an initial layer whose duration is of order $\varepsilon^{\alpha}=\varepsilon^{1-2\gamma}$, so, even if the evaluation M_0 is not accurate, it is quickly improved. We will say that the filter has short memory. If one wants to study the behaviour of the filter for small times, one can use the time change $\overline{t}=t/\varepsilon^{1-2\gamma}$; then, after a normalization, it turns out that the state and observation noises have the same order of magnitude, so we are reduced to the problem of [4].

2. Recall that the estimate (2.19) is uniform as t tends to infinity, though the signal process may have unbounded moments.

3. An immediate consequence of proposition 2.2 is

$$\hat{X}_t-M_t = O(\sqrt{\varepsilon}) \tag{2.26}$$

We will prove that the variance of the filter is exactly of order ε, so the estimate (2.19) is the best possible one. However, (2.26) can be improved; subsequently, we will consider several filters satisfying the assumptions of proposition 2.2, and our purpose will consist of choosing the processes U and V to get better bounds.

4. Let us consider the simple filter

$$dM_t = \frac{\nu}{\varepsilon^{1-2\gamma}}(dY_t - h(M_t)dt) \tag{2.27}$$

Then the equation for X_t-M_t can again be written in the form (2.6) with the same α and ρ_t but with $A_t=\varepsilon^{1-2\gamma}b(X_t)$ and $\zeta_t=\nu(h(X_t)-h(M_t))$. From lemma 2.1, the estimate (2.19) is still valid provided that the moments of $b(X_t)$ are dominated by $\varepsilon^{2\gamma-1/2}$. If we only want to obtain an estimate on bounded time intervals $[t_0,T]$, this condition holds as soon as $\gamma \leq 1/4$; if $\gamma > 1/4$, we can estimate X_t-M_t by $\varepsilon^{1-2\gamma}$. The difference between (2.25) and (2.27) consists of dropping the drift coefficient b. Why can we do this for small γ? Since the filter has short memory, to obtain an estimate at time t, the law of the signal process has not to be considered on the entire time interval $[0,t]$, but only on a small interval before t, and it is well-known that, for uniformly elliptic diffusions studied on a small time interval, the drift coefficient is negligible with respect to the diffusion coefficient. This explains why the filter (2.27) may be acceptable. However, the larger γ is, the longer the filter memory is, the more influential the drift is and therefore if we neglect it, the quality of the filter decreases more quickly as γ increases.

Let us conclude this section with an immediate consequence of lemma 2.1 and proposition 2.2. This result will be used a great deal throughout this work.

Corollary 2.4 *Let M_t be a process satisfying the assumptions of proposition 2.2 and let Z_t be a process such that each moment of Z_0 is dominated by some ε^r and*

$$dZ_t = \frac{1}{\varepsilon^{1-2\gamma}}\left[(-\zeta_t + B_t)dt + C_t(dY_t - h(M_t)dt)\right] \tag{2.28}$$

where ζ_t satisfies the assumption (***) of lemma 2.1 and B_t and C_t are $\underline{\underline{F}}_t$-adapted processes such that

$$B_t = O(\varepsilon^k) \quad \text{and} \quad C_t = O(\varepsilon^{k-\frac{1}{2}}) \tag{2.29}$$

Then $Z_t = O(\varepsilon^k)$.

3. A formula for the a priori density

From now on, the regularity assumptions are (A1). Then it is well-known that for every t, the law of the variable X_t is absolutely continuous with respect to the Lebesgue measure and that its density $p(t,x)$ is solution of the Fokker-Planck equation

$$\dot{p} = \underline{\underline{L}}^* p \tag{3.1}$$

where $\underline{\underline{L}}^*$ is the formal adjoint of $\underline{\underline{L}}$:

$$\underline{\underline{L}}^* p = \varepsilon^{2\gamma}(ap)'' - (bp)' \tag{3.2}$$

Equation (3.2) is a forward partial differential equation, so, after a time reversal, its solution is given by the Feynmann-Kac formula. Actually, the purpose of this section is to derive a more precise representation formula for p. We are going to use the time reversal of the signal X. So fix a time $t>0$, consider the subspace Ω^t of Ω^X made of the continuous functions from $[0,t]$ into $\mathbb{R}$ and define on this subspace

$$\overline{X}_s = X_{t-s} \quad \text{and} \quad \overline{\underline{\underline{F}}}_s(X) = \underline{\underline{F}}_t^{t-s}(X) \tag{3.3}$$

(in order to get readable formulas, since t is fixed, we will not mark the dependence on t in the notations). Define also $\overline{p}(s,x)=p(t-s,x)$.

Proposition 3.1 *Assume (A1). There exists on Ω^t a probability $\overline{P}$ equivalent to the restriction of P^X to Ω^t, such that*

$$d\overline{X}_s = (2\varepsilon^{2\gamma}a'-b)(\overline{X}_s)ds + \varepsilon^\gamma\sigma(\overline{X}_s)d\overline{W}_s \tag{3.4}$$

for some $(\overline{\underline{\underline{F}}}_s(X),\overline{P})$ Brownian motion $\overline{W}_s$, and $\overline{P}(\overline{X}_0\in dx)=p(t,x)dx$. Moreover,

$$\overline{p}(0,\overline{X}_0) = \overline{p}(t,\overline{X}_t)\,\frac{d\overline{P}}{dP^X}\,\exp\int_0^t(\varepsilon^{2\gamma}a''-b')(\overline{X}_s)ds \quad a.s. \tag{3.5}$$

or equivalently,

$$p(t,X_t) = p_0(X_0)\,\frac{d\overline{P}}{dP^X}\,\exp\int_0^t(\varepsilon^{2\gamma}a''-b')(X_s)ds \quad a.s. \tag{3.6}$$

Proof

We prove the proposition in the case $\gamma=0$; it is then sufficient to replace σ by $\varepsilon^\gamma\sigma$ to obtain the general case. The existence of a probability $\overline{P}$ such that (3.4) is satisfied with the initial distribution $p(t,x)dx$ results from the existence of a solution to the martingale problem defined by the generator

$$(2a'-b)(x)\frac{d}{dx} + a(x)\frac{d^2}{dx^2}$$

We have to prove that $\overline{P}$ is equivalent to P^X and that (3.5) holds. First suppose that (A2) is satisfied; then $p(s,x)$ is $C^{1,2}$ and, by applying Itô's formula to the function $\log\overline{p}(s,x)$, one obtains

$$\overline{p}(t,\overline{X}_t) = \overline{p}(0,\overline{X}_0)\exp\left\{\int_0^t\sigma\frac{\overline{p}'}{\overline{p}}(s,\overline{X}_s)d\overline{W}_s \right. \tag{3.7}$$

$$\left. + \int_0^t\left(\frac{\dot{\overline{p}}}{\overline{p}}+(2a'-b)\frac{\overline{p}'}{\overline{p}}+a\frac{\overline{p}''}{\overline{p}}-a\frac{\overline{p}'^2}{\overline{p}^2}\right)(s,\overline{X}_s)ds\right\}$$

and, by the Fokker-Planck equation (3.1),

$$\overline{p}(t,\overline{X}_t) = \overline{p}(0,\overline{X}_0)\exp\left\{\int_0^t \sigma\frac{\overline{p}'}{\overline{p}}(s,\overline{X}_s)d\overline{W}_s - \int_0^t (a''-b'+a\frac{\overline{p}'^2}{\overline{p}^2})(s,\overline{X}_s)ds\right\} \tag{3.8}$$

This formula can then be extended to the set of assumptions (A1). Let P^* be the positive measure defined on Ω^t by

$$\frac{dP^*}{d\overline{P}} = \exp\left\{\int_0^t \sigma\frac{\overline{p}'}{\overline{p}}(s,\overline{X}_s)\overline{W}_s - \int_0^t a\frac{\overline{p}'^2}{\overline{p}^2}(s,\overline{X}_s)ds\right\} \tag{3.9}$$

From (3.8),

$$\frac{dP^*}{d\overline{P}} = \frac{p_0(\overline{X}_t)}{\overline{p}(0,\overline{X}_0)}\exp\int_0^t (a''-b')(\overline{X}_s)ds \tag{3.10}$$

so this density is the value at time t of a positive $\overline{P}$-local martingale with value 1 at time 0, and which is bounded when $\overline{X}_0$ is fixed (because p_0, b' and a'' are bounded). This implies that the local martingale is a martingale with mean value 1, so P^* is a probability. If we prove that P^* is the restriction of P^X to Ω^t then (3.5) will follow immediately from (3.8) and the proposition will be proved. From the Girsanov theorem, there exists a $(\underline{\underline{\overline{F}}}_s(X),P^*)$ Brownian motion W_s^* such that

$$d\overline{X}_s = (2a'-b+2a\frac{\overline{p}'}{\overline{p}})(s,\overline{X}_s)ds + \sigma(\overline{X}_s)dW_s^* \tag{3.11}$$

and $\overline{X}_0$ has law $p(t,x)dx$. On the other hand, from the theory of time reversal of diffusions ([10]), the process

$$W_s^{**} = \int_{t-s}^t (dW_u + \frac{(\sigma p)'}{p}(u,X_u)du) \tag{3.12}$$

is a $(\underline{\underline{\overline{F}}}_s(X),P^X)$ Brownian motion and if one reverses the equation (1.3) of X_s, one obtains (after taking into account the relation between backward and forward Itô integrals) that $\overline{X}_s$ also satisfies equation (3.11) with W^* replaced by W^{**} (in particular, $W_s^*=W_s^{**}$). Thus the process $\overline{X}$ has the same law under P^X and P^* so the two probabilities coincide on $\underline{\underline{\overline{F}}}_t(X)=\underline{\underline{F}}_t(X)$. ■

4. Formulas for the conditional density

Fix a trajectory y of the observation process (such that $y(0)=0$) and define

$$\lambda_t^y \equiv \exp\left\{\frac{1}{\varepsilon^{2-2\gamma}}\left(\int_0^t h(X_s)dy(s)-\frac{1}{2}\int_0^t h^2(X_s)ds\right)\right\} \tag{4.1}$$

where the integral with respect to y is to be understood as a notation for

$$h(X_t)y(t) - \int_0^t y(s)dh(X_s)$$

This integration by parts convention will be used several times in this paper; the integral is also the limit of

$$\int_0^t h(X_s)dy^n(s)$$

where y^n is any sequence of differentiable functions which converge to y ([11]). Write λ_t^y in the form

$$\lambda_t^y = \exp\left\{\frac{1}{\varepsilon^{2-2\gamma}}\left[y(t)h(X_0) + \int_0^t [(y(t)-y(s))\underline{\underline{L}}h(X_s)-\frac{1}{2}h^2(X_s)]ds \right.\right. \tag{4.2}$$

$$\left.\left. + \int_0^t (y(t)-y(s))\sigma h'(X_s)dW_s\right]\right\}$$

From the assumptions, $\underline{\underline{L}}h(x)$ has at most linear growth as $x\to\pm\infty$ and $h^2(x)$ has at least quadratic growth, so the Lebesgue integral is upper bounded. Moreover, $\sigma h'$ is bounded and $\exp(ch(X_0))$ is integrable for every c, so, whenever the trajectories y are uniformly bounded, the moments of λ_t^y are uniformly bounded. Therefore, the Kallianpur-Striebel formula can be written in a robust form as

$$\mathbb{E}[g(X_t)\,|\,\underline{\underline{F}}_t(Y)] = \tilde{\Pi}_t^Y(g)/\tilde{\Pi}_t^Y(1) \quad a.s. \tag{4.3}$$

with

$$\tilde{\Pi}_t^y(g) \equiv \mathbb{E}[g(X_t)\lambda_t^y] \tag{4.4}$$

where the expected value is computed over the space (Ω^X,P^X). Thus an unnormalized conditional density of X_t is given by

$$\tilde{q}(t,x) = \mathbb{E}[\lambda_t^y\,|\,X_t=x]\,p(t,x) = \mathbb{E}[p(t,X_t)\lambda_t^y\,|\,X_t=x] \tag{4.5}$$

We can express $p(t,X_t)$ with equation (3.6); the Radon-Nikodym derivative transforms the P^X-integral into a $\overline{P}$-integral and after the time reversal, we obtain

Proposition 4.1 *Assume (A1). For each observed trajectory y, the unnormalized conditional density of X_t is expressed as*

$$\tilde{q}(t,x) = \overline{\mathbb{E}}[p_0(\overline{X}_t)\lambda_t^y \exp\int_0^t (\varepsilon^{2\gamma}a''-b')(\overline{X}_s)ds \mid \overline{X}_0=x] \tag{4.6}$$

If we note $\overline{y}(s)=y(t)-y(t-s)$, then λ_t^y can be written as

$$\lambda_t^y = \exp\left\{\frac{1}{\varepsilon^{2-2\gamma}}\left(\int_0^t h(\overline{X}_s)d\overline{y}(s)-\frac{1}{2}\int_0^t h^2(\overline{X}_s)ds\right)\right\} \tag{4.7}$$

One can note that one could have derived this formula more directly by using a Feynmann-Kac formula on the Zakai equation, but we will also need the more precise result proved in proposition 3.1. Our goal is the study of the asymptotic behaviour of q as $\varepsilon\to 0$. By means of a Girsanov transformation, we are going to focus the probability around an observable path $m(s)$ which will be chosen in next section; one has to think that, if y were differentiable, the function $m(s)$ would be close to $h^{-1}(\dot{y}(s))$.

Proposition 4.2 *Assume (A1), fix a time $t>0$ and an observed trajectory $y(s)$. Let $m(s)$, $0\le s\le t$, be a bounded measurable function and put $\overline{m}(s)=m(t-s)$. If ε is sufficiently small, one can define on the space Ω^t a probability $\tilde{P}$ such that*

$$\frac{d\tilde{P}}{d\overline{P}} = \exp\left\{-\frac{1}{\varepsilon^{1-\gamma}}\int_0^t (h(\overline{X}_s)-h(\overline{m}(s)))d\overline{W}_s-\frac{1}{2\varepsilon^{2-2\gamma}}\int_0^t (h(\overline{X}_s)-h(\overline{m}(s)))^2ds\right\} \tag{4.8}$$

The process $\overline{X}_s$ is solution of

$$d\overline{X}_s = -\frac{\sigma(\overline{X}_s)}{\varepsilon^{1-2\gamma}}(h(\overline{X}_s)-h(\overline{m}(s)))ds + (2\varepsilon^{2\gamma}a'-b)(\overline{X}_s)ds + \varepsilon^{\gamma}\sigma(\overline{X}_s)d\tilde{W}_s \tag{4.9}$$

with the initial law $p(t,x)dx$, for some $(\overline{F}_{\overline{s}}(X),\tilde{P})$ Brownian motion $\tilde{W}$. Let $\overline{X}_s^x$ be the solution of (4.9) with the initial value $\overline{X}_0^x=x$ and define the functions

$$G(x) = \int_0^x \frac{du}{\sigma(u)} \tag{4.10}$$

$$\overline{F}(s,x) = \int_0^x \frac{h(u)-h(\overline{m}(s))}{\sigma(u)}du \tag{4.11}$$

and the random variable

$$\rho_t^{y,x} = \log p_0(\overline{X}_t^x) + \frac{1}{\varepsilon}\overline{F}(t,\overline{X}_t^x) + \frac{1}{\varepsilon}\int_0^t G(\overline{X}_s^x)dh(\overline{m}(s)) \tag{4.12}$$

$$+ \frac{1}{\varepsilon^{2-2\gamma}}\int_0^t h(\overline{X}_s^x)(d\overline{y}(s)-h(\overline{m}(s))ds)$$

$$+ \int_0^t [\frac{1}{\varepsilon}(h(\overline{X}_s^x)-h(\overline{m}(s))(\frac{b}{\sigma}-\frac{3}{2}\varepsilon^{2\gamma}\sigma')(\overline{X}_s^x)$$

$$- \frac{1}{2\varepsilon^{1-2\gamma}}\sigma h'(\overline{X}_s^x) + (\varepsilon^{2\gamma}a''-b')(\overline{X}_s^x)]ds$$

(where the integrals with respect to $h(\overline{m}(s))$ and $\overline{y}$ are defined by integration by parts or by approximation like λ_t^y). Then the unnormalized conditional density $\tilde{q}$ can be written as

$$\tilde{q}(t,x) = \exp\{\frac{1}{2\varepsilon^{2-2\gamma}}\int_0^t h^2(\overline{m}(s))ds - \frac{1}{\varepsilon}\overline{F}(0,x)\}\ \tilde{\mathbb{E}}[\exp \rho_t^{y,x}] \tag{4.13}$$

Proof

First we have to prove that the measure $\tilde{P}$ defined by (4.8) is a probability if ε is sufficiently small or, equivalently, that the process

$$L_s \equiv \exp\left\{-\frac{1}{\varepsilon^{1-\gamma}}\int_0^s (h(\overline{X}_u)-h(\overline{m}(u)))d\overline{W}_u - \frac{1}{2\varepsilon^{2-2\gamma}}\int_0^s (h(\overline{X}_u)-h(\overline{m}(u)))^2du\right\} \tag{4.14}$$

is a $\overline{P}$ martingale. Define

$$H(x) = \int_0^x \frac{h(u)}{\sigma(u)}du \tag{4.15}$$

From Itô's formula

$$H(\overline{X}_s) = H(\overline{X}_0) + \varepsilon^\gamma\int_0^s h(\overline{X}_u)d\overline{W}_u + \int_0^s (\frac{3}{2}\varepsilon^{2\gamma}h\sigma' + \frac{\varepsilon^{2\gamma}}{2}h'\sigma - \frac{bh}{\sigma})(\overline{X}_u)du \tag{4.16}$$

we deduce

$$L_s = \exp\left\{-\frac{H(\overline{X}_s)-H(\overline{X}_0)}{\varepsilon} + \frac{1}{\varepsilon^{1-\gamma}}\int_0^s h(\overline{m}(u))d\overline{W}_u\right. \tag{4.17}$$

$$\left.+\frac{1}{\varepsilon}\int_0^s[(\frac{3}{2}\varepsilon^{2\gamma}h\sigma'+\frac{\varepsilon^{2\gamma}}{2}h'\sigma-\frac{bh}{\sigma})(\overline{X}_u)-\frac{1}{2\varepsilon^{1-2\gamma}}(h(\overline{X}_u)-h(\overline{m}(u)))^2]du\right\}$$

Fix ε. It is sufficient to prove that, conditionally with respect to $\overline{X}_0$, the process L_s is uniformly integrable. The function H is bounded below and the stochastic integral behaves nicely, so we only have to study the Lebesgue integral. The function inside the first parenthesis has at most quadratic growth and the coefficient of $1/\varepsilon^{1-2\gamma}$ has at least quadratic growth; nevertheless, when ε is sufficiently small, the latter part becomes predominant, so the Lebesgue integral is upper bounded. Thus $\widetilde{P}$ is a probability. Equation (4.9) is simply the Girsanov theorem and it remains to prove (4.13). Let us write the density of $\widetilde{P}$ as

$$\frac{d\widetilde{P}}{d\overline{P}} = \exp\left\{-\frac{1}{\varepsilon}\int_0^t \frac{h(\overline{X}_s)-h(\overline{m}(s))}{\sigma(\overline{X}_s)}d\overline{X}_s\right. \tag{4.18}$$

$$\left.-\frac{1}{2\varepsilon^{2-2\gamma}}\int_0^t(h(\overline{X}_s)-h(\overline{m}(s)))^2ds+\frac{1}{\varepsilon}\int_0^t(h(\overline{X}_s)-h(\overline{m}(s)))(2\varepsilon^{2\gamma}\sigma'-\frac{b}{\sigma})(\overline{X}_s)ds\right\}$$

On the other hand, if $m(s)$ is differentiable, one can apply Itô's formula to the function $\overline{F}$ and obtain

$$\overline{F}(t,\overline{X}_t) = \overline{F}(0,\overline{X}_0) + \int_0^t \frac{h(\overline{X}_s)-h(\overline{m}(s))}{\sigma(\overline{X}_s)}d\overline{X}_s \tag{4.19}$$

$$+\frac{\varepsilon^{2\gamma}}{2}\int_0^t(\sigma h'(\overline{X}_s)-\sigma'(\overline{X}_s))(h(\overline{X}_s)-h(\overline{m}(s)))ds - \int_0^t C(\overline{X}_s)dh(\overline{m}(s))$$

This formula is then extended to all functions m by an approximation argument. From (4.19), we get an expression for the integral with respect to $\overline{X}_s$, we use it in (4.18) to write $d\widetilde{P}/d\overline{P}$, and we multiply by the exponential of (4.12) to obtain

$$\frac{d\widetilde{P}}{d\overline{P}}\exp\rho_t^y = p_0(\overline{X}_t)\,\lambda_t^y\exp\{\frac{1}{\varepsilon}\overline{F}(0,\overline{X}_0)-\frac{1}{2\varepsilon^{2-2\gamma}}\int_0^t h^2(\overline{m}(s))ds \tag{4.20}$$

$$+\int_0^t(\varepsilon^{2\gamma}a''-b')(\overline{X}_s)ds\}$$

where ρ_t^y is $\rho_t^{y,x}$ taken at $x=\overline{X}_0$. Finally, if we transform the $\overline{P}$-integral of (4.6) into a $\widetilde{P}$-integral and use (4.20), we obtain (4.13). ■

Equation (4.13) is not yet quite adapted to the asymptotic study of the filtering problem but we are going to use it to calculate the derivative of q, and then we will be able to obtain asymptotic filters. Since the derivatives of the coefficients of the stochastic differential equation (4.9) are bounded and Holder continuous, we can choose ([8]) a modification of the family $(\overline{X}_s^x, x\in\mathbb{R})$ such that $\overline{X}_s^x$ is almost surely diffentiable with respect to x with derivative satisfying

$$d\left[\frac{\partial\overline{X}_s^x}{\partial x}\right] = \frac{\partial\overline{X}_s^x}{\partial x}\Big[-\frac{1}{\varepsilon^{1-2\gamma}}(\sigma h'(\overline{X}_s^x)+\sigma'(\overline{X}_s^x)(h(\overline{X}_s^x)-h(\overline{m}(s))))ds \tag{4.21}$$

$$+(2\varepsilon^{2\gamma}a''-b')(\overline{X}_s^x)ds+\varepsilon^{\gamma}\sigma'(\overline{X}_s^x)d\widetilde{W}_s\Big]$$

It is easy to prove from this relation that

$$\frac{\partial}{\partial s}\left[\frac{1}{\sigma(\overline{X}_s^x)}\frac{\partial\overline{X}_s^x}{\partial x}\right] = \frac{\partial\overline{X}_s^x}{\partial x}\left(-\frac{h'}{\varepsilon^{1-2\gamma}}+\frac{3}{2}\varepsilon^{2\gamma}\sigma''+\frac{b\sigma'-b'\sigma}{\sigma^2}\right)(\overline{X}_s^x) \tag{4.22}$$

We immediately deduce that, if we define the functional

$$\overline{\Gamma}_{s,t}^{(1),x} = \frac{\sigma(\overline{X}_s^x)}{\sigma(x)}\exp\left\{-\frac{1}{\varepsilon^{1-2\gamma}}\int_0^s\sigma h'(\overline{X}_u^x)du+\int_0^s\left(\frac{3}{2}\varepsilon^{2\gamma}\sigma\sigma''+b\frac{\sigma'}{\sigma}-b'\right)(\overline{X}_u^x)\right\} \tag{4.23}$$

(where there is no more $\widetilde{W}$), then

$$\frac{\partial\overline{X}_s^x}{\partial x} = \overline{\Gamma}_{s,t}^{(1),x} \tag{4.24}$$

To study the differentiability of the variable $\rho_t^{y,x}$, we write (4.12) with the convention of the integration by parts, so we are reduced to study the regularity of integrals depending on parameters; from our regularity assumptions, we obtain that $\rho_t^{y,x}$ is differentiable with respect to x and its derivative is

$$\rho_t^{(1),y,x} = \left(\frac{p_0'}{p_0}(\overline{X}_t^x) + \frac{1}{\varepsilon}\frac{h(\overline{X}_t^x)-h(\overline{m}(t))}{\sigma(\overline{X}_t^x)}\right)\overline{\Gamma}_{0,t}^{(1),x} \tag{4.25}$$

$$+ \frac{1}{\varepsilon^{2-2\gamma}}\int_0^t h'(\overline{X}_s^x)\overline{\Gamma}_{s,t}^{(1),x}(d\overline{y}(s)-\overline{m}(s)ds) + \frac{1}{\varepsilon}\int_0^t \frac{1}{\sigma(\overline{X}_s^x)}\overline{\Gamma}_{s,t}^{(1),x}dh(\overline{m}(s))$$

$$+ \int_0^t (\varepsilon^{2\gamma}a^{(3)}-b'')(\overline{X}_s^x)\overline{\Gamma}_{s,t}^{(1),x}ds + \frac{1}{\varepsilon}\int_0^t \left(\left(\frac{b}{\sigma}-2\varepsilon^{2\gamma}\sigma'\right)h'(\overline{X}_s^x)\right.$$

$$\left.+(h(\overline{X}_s^x)-h(\overline{m}(s)))\left(\left(\frac{b}{\sigma}\right)'-\frac{3}{2}\varepsilon^{2\gamma}\sigma''\right)(\overline{X}_s^x)\right)\overline{\Gamma}_{s,t}^{(1),x}ds$$

From this equation, we are going to prove

Lemma 4.3 *Assume (A1). For every $t>0$, the conditional density $q(t,.)$ of X_t is a.s. differentiable and its derivative is given by*

$$\frac{q'(t,x)}{q(t,x)} = -\frac{1}{\varepsilon}\frac{h(x)-h(\overline{m}(0))}{\sigma(x)} + \frac{\widetilde{\mathbb{E}}[\rho_t^{(1),y,x}\exp\rho_t^{y,x}]}{\widetilde{\mathbb{E}}[\exp\rho_t^{y,x}]} \tag{4.26}$$

Proof

Since

$$\frac{\partial}{\partial x}\exp\rho_t^{y,x} = \rho_t^{(1),y,x}\exp\rho_t^{y,x} \tag{4.27}$$

and $\tilde{q}'/\tilde{q} = q'/q$, the lemma consists of integrating the two sides of (4.27) with respect to $\tilde{P}$ and inverting the $\tilde{P}$-integral and the x-derivative to obtain

$$\frac{\partial}{\partial x}\widetilde{\mathbb{E}}[\exp\rho_t^{y,x}] = \widetilde{\mathbb{E}}[\rho_t^{(1),y,x}\exp\rho_t^{y,x}] \tag{4.28}$$

Fix ε. To justify this derivation, it is sufficient to prove that the family of random variables

$$\frac{1}{\overline{x}-x}\int_x^{\overline{x}}\rho_t^{(1),y,z}\exp\rho_t^{y,z}dz$$

is $\tilde{P}$-uniformly integrable as $\overline{x}\to x$. This property will hold if we prove that, for any bounded subset K of $\mathbb{R}$, the family $\{\widetilde{\mathbb{E}}[|\rho_t^{(1),y,x}\exp\rho_t^{y,x}|^r], x\in K\}$ is bounded for some $r>1$. One can prove that all the moments of $\rho_t^{(1),y,x}$ are bounded, so we have to estimate $\widetilde{\mathbb{E}}[|\exp\rho_t^{y,x}|^r]$. We use equation (4.20) to write, for $x\in K$,

$$\tilde{\mathbb{E}}[\,|\exp\rho_t^{y,x}|^r\,] \le C_K\ \tilde{\mathbb{E}}[(\lambda_t^y \frac{d\overline{P}}{d\tilde{P}})^r \mid \overline{X}_0=x\,] \tag{4.29}$$

(here, ε is fixed, the constants may depend on it). Then one can prove that all the moments of λ_t^y conditioned by $\overline{X}_0=x$ are bounded so it is sufficient to estimate

$$\tilde{\mathbb{E}}[(\frac{d\overline{P}}{d\tilde{P}})^r \mid \overline{X}_0=x\,] = \overline{\mathbb{E}}[(\frac{d\overline{P}}{d\tilde{P}})^{r-1} \mid \overline{X}_0=x\,] \tag{4.30}$$

From (4.17) and $d\overline{P}/d\tilde{P}=L_t^{-1}$, it follows that

$$\frac{d\overline{P}}{d\tilde{P}} \le C_1 \exp\left\{C_2 \sup_{0\le s\le t} \overline{X}_s^2 - \frac{1}{\varepsilon^{1-\gamma}}\int_0^t h(\overline{m}(s))\,d\overline{W}_s\right\} \tag{4.31}$$

so

$$\overline{\mathbb{E}}[(\frac{d\overline{P}}{d\tilde{P}})^{r-1} \mid \overline{X}_0=x\,] \le C_3\ \overline{\mathbb{E}}[\exp\{C_4(r-1)\sup_{0\le s\le t}\overline{X}_s^2\} \mid \overline{X}_0=x\,] \tag{4.32}$$

From estimates on the moments of diffusion processes ([6]), the right side of (4.32) is finite if $(r-1)$ is sufficiently small, and actually, one can prove that it is uniformly bounded when x stays in K. ■

Now do not fix any more $\overline{X}_0=x$; define the variables $\overline{\Gamma}_{s,t}^{(1)}$, ρ_t^y, $\rho_t^{(1),y}$ by replacing x by $\overline{X}_0$ in the corresponding variables with superscript x. Then

$$\frac{q'}{q}(t,\overline{X}_0) = -\frac{1}{\varepsilon}\frac{h(\overline{X}_0)-h(\overline{m}(0))}{\sigma(\overline{X}_0)} + \frac{\tilde{\mathbb{E}}[\rho_t^{(1),y}\exp\rho_t^y \mid \overline{X}_0]}{\tilde{\mathbb{E}}[\exp\rho_t^y \mid \overline{X}_0]} \tag{4.33}$$

Moreover, from (3.5) and (4.20),

$$\exp\rho_t^y = \frac{dP^X}{d\tilde{P}}\ \overline{p}(0,\overline{X}_0)\ \lambda_t^y \exp\{\frac{1}{\varepsilon}\overline{F}(0,\overline{X}_0) - \frac{1}{2\varepsilon^{2-2\gamma}}\int_0^t h^2(\overline{m}(s))\,ds\} \tag{4.34}$$

so, since the law of $\overline{X}_0$ is not affected by the changes of probability laws, we derive after simplifying the ratio in (4.33) and reversing the time

$$\frac{q'}{q}(t,X_t) = -\frac{1}{\varepsilon}\frac{h(X_t)-h(m(t))}{\sigma(X_t)} + \frac{\mathbb{E}[\rho_t^{(1),y}\lambda_t^y \mid X_t]}{\mathbb{E}[\lambda_t^y \mid X_t]} \tag{4.35}$$

Now, by the Kallianpur-Striebel formula, the last ratio is simply $\mathbb{E}[\rho_t^{(1),Y} \mid X_t\,, Y=y\,]$ and, since we have reversed the time, we write $\rho_t^{(1),y}$ by means of the variable $\Gamma_{s,t}^{(1)}=\overline{\Gamma}_{t-s,t}^{(1)}$ as

$$\rho_t^{(1),y} = (\frac{p_0'}{p_0}(X_0)+\frac{1}{\varepsilon}\frac{h(X_0)-h(m(0))}{\sigma(X_0)})\Gamma_{0,t}^{(1)} \tag{4.36}$$

$$+\frac{1}{\varepsilon^{2-2\gamma}}\int_0^t h'(X_s)\Gamma_{s,t}^{(1)}(dy(s)-h(m(s))ds)-\frac{1}{\varepsilon}\int_0^t\frac{1}{\sigma(X_s)}\Gamma_{s,t}^{(1)}dh(m(s))$$

$$+\int_0^t(\varepsilon^{2\gamma}a^{(3)}-b'')(X_s)\Gamma_{s,t}^{(1)}ds+\frac{1}{\varepsilon}\int_0^t[(\frac{b}{\sigma}-2\varepsilon^{2\gamma}\sigma')h'(X_s)$$

$$+(h(X_s)-h(m(s)))((\frac{b}{\sigma})'-\frac{3}{2}\varepsilon^{2\gamma}\sigma'')(X_s)]\Gamma_{s,t}^{(1)}ds$$

Now if we fix no more the trajectory of Y, then the function m becomes an $\underline{\underline{F}}_s(Y)$-adapted process M_s. If we assume that M_s is a $\underline{\underline{F}}_s(Y)$-semimartingale, then the integrals with respect to $y(s)$ and $h(m(s))$ become usual Itô integrals with respect to Y_s and $h(M_s)$ (for each fixed t, the integrated processes are $\underline{\underline{F}}_t(X)\vee\underline{\underline{F}}_s(Y)$-adapted), so equations (4.23), (4.35) and (4.36) imply

Proposition 4.4 *Assume (A1). Then the normalized conditional density $q(t,x)$ of X_t is a.s. differentiable with respect to x and its derivative satisfies, for any $\underline{\underline{F}}_t(Y)$-semimartingale M_t,*

$$\frac{q'}{q}(t,X_t) = -\frac{1}{\varepsilon}\frac{h(X_t)-h(M_t)}{\sigma(X_t)}+\mathbb{E}[\rho_t^{(1)}\mid X_t\,.\,\underline{\underline{F}}_t(Y)] \tag{4.37}$$

with

$$\rho_t^{(1)} = (\frac{p_0'}{p_0}(X_0)+\frac{1}{\varepsilon}\frac{h(X_0)-h(M_0)}{\sigma(X_0)})\Gamma_{0,t}^{(1)} \tag{4.38}$$

$$+\frac{1}{\varepsilon^{2-2\gamma}}\int_0^t h'(X_s)\Gamma_{s,t}^{(1)}(dY_s-h(M_s)ds)-\frac{1}{\varepsilon}\int_0^t\frac{1}{\sigma(X_s)}\Gamma_{s,t}^{(1)}dh(M_s)$$

$$+\int_0^t(\varepsilon^{2\gamma}a^{(3)}-b'')(X_s)\Gamma_{s,t}^{(1)}ds+\frac{1}{\varepsilon}\int_0^t[(\frac{b}{\sigma}-2\varepsilon^{2\gamma}\sigma')h'(X_s)$$

$$+(h(X_s)-h(M_s))((\frac{b}{\sigma})'-\frac{3}{2}\varepsilon^{2\gamma}\sigma'')(X_s)]\Gamma_{s,t}^{(1)}ds$$

and

$$\Gamma_{s,t}^{(1)} = \frac{\sigma(X_s)}{\sigma(X_t)}\exp\left\{-\frac{1}{\varepsilon^{1-2\gamma}}\int_s^t\sigma h'(X_u)du+\int_s^t(\frac{3}{2}\varepsilon^{2\gamma}\sigma\sigma''+b\frac{\sigma'}{\sigma}-b')(X_u)du\right\} \tag{4.39}$$

5. A one-dimensional filter

In previous section, the process M_t appears as a control process and we have now to choose it; we want the last term of (4.37) to be negligible with respect to $1/\varepsilon$; if we define

$$F(t,x) = \int_0^x \frac{h(u)-h(M_t)}{\sigma(u)} du$$

the density $q(t,x)$ will then behave like the normalization of $\exp\{-F(t,x)/\varepsilon\}$. This function has a unique maximum at $x=M_t$ and is very small outside any neighbourhood of M_t (if h is linear and σ is constant, this is the density of the normal law with mean M_t and variance $\varepsilon\sigma/h'$). Therefore M_t will be a good approximation for the optimal filter $\hat{X}_t$. We are going to take for M_t the solution of a differential equation of the type described in section 2 and we have to choose U and V. As in remark 4 of section 2, the smaller γ will be, the better the approximate filter will be.

Theorem 5.1 *Assume (A1). Let m_0 be a real number and let M_t be the solution of the equation*

$$dM_t = b(M_t)dt + \frac{\sigma(M_t)}{\varepsilon^{1-2\gamma}}(dY_t - h(M_t)dt) \tag{5.1}$$

with the initial condition $M_0=m_0$. Then

$$\hat{X}_t - M_t = O(\varepsilon \vee \varepsilon^{\frac{3}{2}-2\gamma}) \tag{5.2}$$

Proof

Since σ is bounded below, we can apply proposition 2.2 and we obtain a first estimate $X_t-M_t=O(\sqrt{\varepsilon})$. We can also choose ε_0 sufficiently small so that

$$C \equiv \inf_x \left[\sigma h' - \frac{3}{2}\varepsilon_0|\sigma\sigma''| - \varepsilon_0^{1-2\gamma}\sigma|(\frac{b}{\sigma})'|\right](x) > 0 \tag{5.3}$$

If we note c_1 and c_2 the lower and upper bounds of σ, from (4.39),

$$\Gamma_{s,t}^{(1)} \leq \frac{c_2}{c_1}\exp\left\{-\frac{C(t-s)}{\varepsilon^{1-2\gamma}}\right\} \tag{5.4}$$

if $\varepsilon \leqq \varepsilon_0$. On the other hand, after remarking that

$$dh(M_s) = (bh' + \varepsilon^{2\gamma} ah'')(M_s)ds + \frac{\sigma h'(M_s)}{\varepsilon^{1-2\gamma}}(dY_s - h(M_s)ds) \tag{5.5}$$

one can write (4.38) in the form

$$\rho_t^{(1)} = \psi_1(X_0,M_0)\Gamma_{0,t}^{(1)} + \frac{1}{\varepsilon^{1-2\gamma}}\int_0^t \psi_2(X_s,M_s)\Gamma_{s,t}^{(1)}ds \tag{5.6}$$

$$+ \frac{1}{\varepsilon^{2-2\gamma}}\int_0^t \psi_3(X_s,M_s)\Gamma_{s,t}^{(1)}(dY_s - h(M_s)ds)$$

with the notations

$$\psi_1(x,m) = \frac{p_0'}{p_0}(x) + \frac{1}{\varepsilon}\frac{h(x)-h(m)}{\sigma(x)} \tag{5.7}$$

$$\psi_2(x,m) = -\frac{ah''(m)}{\sigma(x)} + (\varepsilon a^{(3)} - \varepsilon^{1-2\gamma}b'' - 2\sigma' h')(x) \tag{5.8}$$

$$+ (h(x)-h(m))\left(\frac{1}{\varepsilon^{2\gamma}}\left(\frac{b}{\sigma}\right)' - \frac{3}{2}\sigma''\right)(x) + \frac{bh'(x)-bh'(m)}{\varepsilon^{2\gamma}\sigma(x)}$$

$$\psi_3(x,m) = \frac{\sigma h'(x) - \sigma h'(m)}{\sigma(x)} \tag{5.9}$$

Now, we estimate these terms. The moments of $\psi_1(X_0,M_0)$ are dominated by $1/\varepsilon$, so from (5.4),

$$\psi_1(X_0,M_0)\Gamma_{0,t}^{(1)} = O(\varepsilon^k) \tag{5.10}$$

for any k. To study the integral involving ψ_2, we cut the integral over $[0,t]$ into an integral over $[0,s_0]$ which is negligible because $\Gamma_{s,t}$ is very small, and an integral over $[s_0,t]$; on this interval, we remark that since h and bh' are Lipschitz, the process $\psi_2(X_s,M_s)$ is dominated by $1\vee\varepsilon^{\frac{1}{2}-2\gamma}$. Therefore we prove from (5.4) that

$$\frac{1}{\varepsilon^{1-2\gamma}}\int_0^t \psi_2(X_s,M_s)\Gamma_{s,t}^{(1)}ds = O(1\vee\varepsilon^{\frac{1}{2}-2\gamma}) \quad . \tag{5.11}$$

Now, let us consider the last term which is to be studied:

$$\frac{1}{\varepsilon^{2-2\gamma}}\int_0^t \frac{\sigma h'(X_s)-\sigma h'(M_s)}{\sigma(X_s)}\Gamma_{s,t}^{(1)}(dY_s - h(M_s)ds) \tag{5.12}$$

$$= \frac{1}{\varepsilon^{2-2\gamma}}\int_0^t \frac{1}{\sigma(X_s)}(\sigma h'(X_s)-\sigma h'(M_s))(h(X_s)-h(M_s))\Gamma_{s,t}^{(1)}ds$$

$$+ \frac{1}{\varepsilon^{1-\gamma}}\int_0^t \frac{\sigma h'(X_s)-\sigma h'(M_s)}{\sigma(X_s)}\Gamma_{s,t}^{(1)}dB_s$$

$$= O(1)$$

This last estimate is proved like (5.11) and by estimating the L^q norm of the stochastic integral by the L^q norm of the square root of its quadratic variation. By summing up (5.10), (5.11), (5.12), we derive

$$\rho_t^{(1)} = O(1\vee\varepsilon^{\frac{1}{2}-2\gamma}) \tag{5.13}$$

After taking the expected value of this equation conditioned by $X_t \vee \underline{\underline{F}}_t(Y)$, we deduce from proposition 4.4

$$\frac{q'}{q}(t,X_t) + \frac{1}{\varepsilon}\frac{(h(X_t)-h(M_t))}{\sigma(X_t)} = O(1\vee\varepsilon^{\frac{1}{2}-2\gamma}) \tag{5.14}$$

so, by multiplying by $\sigma(X_t)$ and conditioning by $\underline{\underline{F}}_t(Y)$,

$$\int_{-\infty}^{+\infty}(\sigma\frac{q'}{q}(t,x) + \frac{1}{\varepsilon}(h(x)-h(M_t)))q(t,x)dx = O(1\vee\varepsilon^{\frac{1}{2}-2\gamma}) \tag{5.15}$$

where we have also deduced from (5.14) the (almost sure) integrability of the function inside the integral. Thus the function $\sigma q'$ is almost surely Lebesgue integrable with respect to x, as well as $\sigma' q$ (since σ' is bounded) and $(\sigma q)'$ (by addition of the two previous functions). But the integral of $(\sigma q)'$ is necessarily 0 because, as $x\to\pm\infty$, the only possible limit of $\sigma q(t,x)$ is 0. So, by the integration by parts formula,

$$\int_{-\infty}^{+\infty}\sigma q'(t,x)dx = -\int_{-\infty}^{+\infty}\sigma' q(t,x)dx = O(1) \tag{5.16}$$

and consequently, from (5.15) and (5.16),

$$\hat{h}_t - h(M_t) = O(\varepsilon\vee\varepsilon^{\frac{3}{2}-2\gamma}) \tag{5.17}$$

By developing the function h around M_t and since $(X_t - M_t)^2 = O(\varepsilon)$, we obtain

$$\hat{h}_t = h(M_t) + h'(M_t)(\hat{X}_t - M_t) + O(\varepsilon) \tag{5.18}$$

so since h' is uniformly positive, (5.2) is proved from (5.17) and (5.18). ■

Remarks

1. If $\gamma \leq 1/4$, the approximation error is of order ε and, as $\gamma \to 1/2$, this error order tends to $\sqrt{\varepsilon}$.

2. As in section 2, we can realize the growing influence of the drift as γ increases by considering the filter

$$dM_t = \frac{\sigma(M_t)}{\varepsilon^{1-2\gamma}}(dY_t - h(M_t)dt) \tag{5.19}$$

Restricting ourselves to bounded time intervals, or assuming that the moments of $b(X_t)$ are bounded, we can prove that

$$\hat{X}_t - M_t = O(\varepsilon \vee \varepsilon^{1-2\gamma}) \tag{5.20}$$

so this filter is worse than (5.1) as soon as $\gamma > 0$.

3. Consider the case h linear and σ constant (i.e. the only nonlinearities are b and the initial density p_0). Then one can improve the estimate (5.2). Indeed,

$$\psi_2(x,m) = -\varepsilon^{1-2\gamma} b''(x) + \varepsilon^{-2\gamma}\frac{h'}{\sigma}[(x-m)b'(x) + b(x) - b(m)] \tag{5.21}$$

so the estimate (5.11) can be replaced by $O(\varepsilon^{\frac{1}{2}-2\gamma})$; moreover, ψ_3 and the left side of (5.16) are zero. Finally, one obtains $\hat{X}_t - M_t = O(\varepsilon^{\frac{3}{2}-2\gamma})$. For the filter (5.19) the error bound is $O(\varepsilon^{1-2\gamma})$.

We will need a slight generalisation of theorem 5.1; the proof of this result is similar.

Theorem 5.2 *Assume (A1). Let M_t be a $\underline{\underline{F}}_t(Y)$-adapted process such that M_t satisfies an equation of type (2.18) with $V_t \geq c > 0$ and*

$$U_t = O(\varepsilon^{1/2} \vee \varepsilon^{2\gamma}) \quad \text{and} \quad V_t = \sigma(M_t) + O(\varepsilon^{1/2} \vee \varepsilon^{1-2\gamma}) \tag{5.22}$$

Then the estimate (5.2) is still valid. Moreover, the condition $V_t \geq c > 0$ can be *dropped if we know that* $X_t - M_t = O(\sqrt{\varepsilon})$.

6. Conditional moments

To improve the approximation described in previous section, we can increase the dimension of the approximate filtering equation; a classical method to find two-dimensional filters is to approximate the differential equation satisfied by the vector $(\widehat{X}_t, \widehat{e_t^2})$ with $e_t = X_t - \widehat{X}_t$. This is generally done by arbitrarily replacing the conditional moments $\widehat{e_t^n}$, $n \geq 3$, by some values. Then, if one wants to estimate the approximation error, one has to prove that the conditional moments were correctly chosen. The purpose of this section is the evaluation of these moments. In previous section, we have used the first spatial derivative of the conditional density; in this section, we will need further derivatives.

First, let us define a notation. If g^i, $1 \leq i \leq n$, are n functions defined on $\mathbb{R}$, we will note

$$e_t^{g^1 \cdots g^n} = \prod_{i=1}^{n} (g^i(X_t) - \widehat{g_t^i}) \tag{6.1}$$

and $\widehat{e_t}^{g^1 \cdots g^n}$ will be the conditional expectation of this variable. The identity function will be noted by X, so for instance, the second conditional moment $\widehat{e_t^2}$ will be noted from now on $\widehat{e}_t^{XX}$.

Proposition 6.1 *Assume (A2). Then*

$$\widehat{e}_t^{hh} = \varepsilon \sigma h'(\widehat{X}_t) + O(\varepsilon^{3/2} \vee \varepsilon^{2-2\gamma}) \tag{6.2}$$

Proof

Let M_t be the solution of (5.1) and define

$$\varphi(t,x) \equiv q(t,x) \exp\left(\frac{1}{\varepsilon} \int_0^x \frac{h(u) - h(M_t)}{\sigma(u)} du\right) \tag{6.3}$$

Note that equation (5.14) can be written as

$$\frac{\varphi'}{\varphi}(t, X_t) = O(1 \vee \varepsilon^{\frac{1}{2} - 2\gamma}) \tag{6.4}$$

and let us now study further derivatives. Return to the probability $\widetilde{P}$ and the reverse process $\overline{X}_s^x$ of section 4. If we fix a trajectory y of Y, lemma 4.3 can be written as

$$\frac{\varphi'}{\varphi}(t,x)=\frac{\widetilde{\mathbb{E}}[\rho_t^{(1),y,x}\exp\rho_t^{y,x}]}{\widetilde{\mathbb{E}}[\exp\rho_t^{y,x}]} \tag{6.5}$$

From (A2), $\overline{X}_s^x$ is twice differentiable with respect to x and from (4.23), (4.24),

$$\frac{\partial^2\overline{X}_s^x}{\partial x^2}=\overline{\Gamma}_{s,t}^{(2),x} \tag{6.6}$$

with

$$\overline{\Gamma}_{s,t}^{(2),x}\equiv(\frac{\sigma'}{\sigma})(\overline{X}_s^x)(\overline{\Gamma}_{s,t}^{(1),x})^2-\frac{\sigma'}{\sigma}(x)\overline{\Gamma}_{s,t}^{(1),x} \tag{6.7}$$

$$+\overline{\Gamma}_{s,t}^{(1),x}\int_0^s(-\frac{1}{\varepsilon^{1-2\gamma}}(\sigma h')'+(\frac{3}{2}\varepsilon^{2\gamma}\sigma\sigma''+b\frac{\sigma'}{\sigma}-b'))(\overline{X}_u^x)\overline{\Gamma}_{u,t}^{(1),x}du$$

If we differentiate (4.25), we obtain that the second derivative of $\rho_t^{y,x}$ is

$$\rho_t^{(2),y,x}\equiv\frac{\partial\psi_1}{\partial x}(\overline{X}_t^x,\overline{m}(t))(\overline{\Gamma}_{0,t}^{(1),x})^2+\frac{1}{\varepsilon^{1-2\gamma}}\int_0^t\frac{\partial\psi_2}{\partial x}(\overline{X}_s^x,\overline{m}(s))(\overline{\Gamma}_{s,t}^{(1),x})^2ds \tag{6.8}$$

$$+\frac{1}{\varepsilon^{2-2\gamma}}\int_0^t\frac{\partial\psi_3}{\partial x}(\overline{X}_s^x,\overline{m}(s))(\overline{\Gamma}_{s,t}^{(1),x})^2(d\overline{y}(s)-h(\overline{m}(s))ds)$$

$$+\psi_1(\overline{X}_t^x,\overline{m}(t))\overline{\Gamma}_{0,t}^{(2),x}+\frac{1}{\varepsilon^{1-2\gamma}}\int_0^t\psi_2(\overline{X}_s^x,\overline{m}(s))\overline{\Gamma}_{s,t}^{(2),x}ds$$

$$+\frac{1}{\varepsilon^{2-2\gamma}}\int_0^t\psi_3(\overline{X}_s^x,\overline{m}(s))\overline{\Gamma}_{s,t}^{(2),x}(d\overline{y}(s)-h(\overline{m}(s))ds)$$

(remember that the functions ψ_i were defined by (5.7)-(5.9)). As in section 5, one can justify the differentiation of equation (6.5) and obtain

$$\frac{\varphi''}{\varphi}(t,x)=\frac{1}{\widetilde{\mathbb{E}}[\exp\rho_t^{y,x}]}\widetilde{\mathbb{E}}[[(\rho_t^{(1),y,x})^2+\rho_t^{(2),y,x}]\exp\rho_t^{y,x}] \tag{6.9}$$

So, if we do not fix x and y and if we return to the actual probability P,

$$\frac{\varphi''}{\varphi}(t,X_t)=\mathbb{E}[(\rho_t^{(1)})^2+\rho_t^{(2)}\mid \underline{\underline{F}}_t(Y),X_t] \tag{6.10}$$

In previous section, we have estimated $\rho_t^{(1)}$. We can proceed similarly for the process $\rho_t^{(2)}$ given by (6.8): from (6.7), the variable $\Gamma_{s,t}^{(2)}=\overline{\Gamma}_{t-s,t}^{(2)}$ satisfies an inequality similar to (5.4); we can compute the derivatives of ψ_1, ψ_2 and ψ_3 and prove that they are dominated respectively by $1/\varepsilon$, $1 \vee \varepsilon^{-2\gamma}$ and 1. This implies that

$$\frac{\varphi''}{\varphi}(t,X_t) = O(\varepsilon^{-1/2} \vee \varepsilon^{-2\gamma}) \tag{6.11}$$

Therefore, from (6.4) and (6.11),

$$\frac{1}{\varphi}(\sigma(\sigma\varphi)')'(t,X_t) = O(\varepsilon^{-1/2} \vee \varepsilon^{-2\gamma}) \tag{6.12}$$

so, by taking the conditional expectation,

$$\int_{-\infty}^{+\infty} (\sigma(\sigma\varphi)')' \frac{q}{\varphi}(t,x)dx = O(\varepsilon^{-1/2} \vee \varepsilon^{-2\gamma}) \tag{6.13}$$

and after two integrations by parts,

$$\int_{-\infty}^{+\infty} \varphi\sigma(\sigma(\frac{q}{\varphi})')'(t,x)dx = O(\varepsilon^{-1/2} \vee \varepsilon^{-2\gamma}) \tag{6.14}$$

(these integrations by parts are justified like in proof of theorem 5.1). Now,

$$\sigma(\sigma(\frac{q}{\varphi})')'(t,x) = (-\frac{1}{\varepsilon}\sigma h'(x) + \frac{1}{\varepsilon^2}(h(x)-h(m))^2)\ \frac{q}{\varphi}(t,x) \tag{6.15}$$

so, by using (6.15) in (6.14),

$$\mathbb{E}[-\frac{1}{\varepsilon}\sigma h'(X_t) + \frac{1}{\varepsilon^2}(h(X_t)-h(M_t))^2 | \underline{\underline{F}}_t(Y)] = O(\varepsilon^{-1/2} \vee \varepsilon^{-2\gamma}) \tag{6.16}$$

Since $X_t - \widehat{X}_t$ is of order $\sqrt{\varepsilon}$,

$$\mathbb{E}[(h(X_t)-h(M_t))^2 | \underline{\underline{F}}_t(Y)] = \varepsilon\sigma h'(\widehat{X}_t) + O(\varepsilon^{3/2} \vee \varepsilon^{2-2\gamma}) \tag{6.17}$$

On the other hand,

$$\mathbb{E}[(h(X_t)-h(M_t))^2 | \underline{\underline{F}}_t(Y)] - \mathbb{E}[(h(X_t)-\widehat{h}_t)^2 | \underline{\underline{F}}_t(Y)] \tag{6.18}$$

$$= (h(M_t)-\widehat{h}_t)^2 = O(\varepsilon^2 \vee \varepsilon^{3-4\gamma})$$

from (5.17). This completes the proof of the proposition. ■

Corollary 6.2 *Assume (A2). Then*

$$\widehat{e}_t^{XX} = \varepsilon\frac{\sigma}{h'}(\widehat{X}_t) + O(\varepsilon^{3/2} \vee \varepsilon^{2-2\gamma}) \tag{6.19}$$

and more generally, if g_1 and g_2 are functions of $\underline{B}(2)$, then

$$\widehat{e}_t^{g_1 g_2} = \varepsilon \frac{\sigma g_1' g_2'}{h'}(\widehat{X}_t) + O(\varepsilon^{3/2} \vee \varepsilon^{2-2\gamma}) \tag{6.20}$$

Proof

It is sufficient to use theorem 6.1 and to remark

$$\begin{aligned} h(X_t) - \widehat{h}_t &= h(X_t) - h(\widehat{X}_t) + O(\varepsilon) \\ &= h'(\widehat{X}_t)(X_t - \widehat{X}_t) + O(\varepsilon) \end{aligned} \tag{6.21}$$

to check (6.19). For (6.20), we use similar relations for g_1 and g_2. ■

Formula (6.19) provides us with an estimate of the second conditional moment announced in (1.9) but the evaluation $\widehat{e}_t^{hX} \approx \varepsilon\sigma(\widehat{X}_t)$ is more interesting for finding approximate expressions for $\widehat{X}_t$. Indeed, $\widehat{X}_t$ is solution of the differential equation

$$d\widehat{X}_t = \widehat{b}_t\, dt + \frac{\widehat{e}_t^{hX}}{\varepsilon^{2-2\gamma}}(dY_t - \widehat{h}_t\, dt) \tag{6.22}$$

The choice of equation (5.1) for the approximate filter is therefore a posteriori justified by the estimate of $\widehat{e}_t^{hX}$.

Note that in proposition 6.1, the control process M_t is only a tool in the proof but is not involved in the result. As in section 5, one can prove that one could have replaced it by any process satisfying the conditions of theorem 5.2. Let us consider $M_t = h^{-1}(\widehat{h}_t)$: then we immediately have $M_t - \widehat{X}_t = O(\varepsilon)$ so $X_t - M_t = O(\sqrt{\varepsilon})$ and the condition $V_t \geq c > 0$ of theorem 5.2 can be dropped; if we write the differential of M_t in the form (2.18) then U_t is easily estimated by $\varepsilon^{1/2} \vee \varepsilon^{2\gamma}$ and

$$V_t = \frac{\widehat{e}_t^{hh}}{\varepsilon h'(M_t)} \tag{6.23}$$

so the condition (5.22) on V is equivalent to proposition 6.1; thus we cannot use directly this process to prove the proposition but we can check with this method that the estimates (6.4) and (6.11) hold for $M_t = h^{-1}(\widehat{h}_t)$. So, from now on, let us fix this process. Assuming (A3), we can derive (6.8) to obtain a process $\rho_t^{(3),y,x}$ and then compute the third derivative of φ by differentiating (6.9):

$$\frac{\varphi^{(3)}}{\varphi}(t,X_t) = \mathbb{E}[(\rho_t^{(1)})^3 + 3\rho_t^{(2)}\rho_t^{(1)} + \rho_t^{(3)} \mid \underline{\underline{F}}_t(Y), X_t] \tag{6.24}$$

The same type of calculation used previously yields

$$\frac{\varphi^{(3)}}{\varphi}(t,X_t) = O(\varepsilon^{-1/2} \vee \varepsilon^{\frac{1}{2}-4\gamma}) \tag{6.25}$$

With this result, we can estimate the third and fourth conditional moments.

Proposition 6.3 *Assume (A3). Then*

$$\widehat{e}_t^{hhh} = 2\varepsilon^2\sigma(\sigma h')'(\widehat{X}_t) + O(\varepsilon^{5/2} \vee \varepsilon^{\frac{7}{2}-4\gamma}) \tag{6.26}$$

$$\widehat{e}_t^{hhhh} = 3\varepsilon^2\sigma^2 h'^2(\widehat{X}_t) + O(\varepsilon^{5/2} \vee \varepsilon^{3-2\gamma}) \tag{6.27}$$

Proof

Let us first study the third moment. From (6.25),

$$\frac{1}{\varphi}(\sigma(\sigma(\sigma\varphi)')')'(t,X_t) = O(\varepsilon^{-1/2} \vee \varepsilon^{\frac{1}{2}-4\gamma}) \tag{6.28}$$

so, after taking the conditional expectation with respect to $\underline{\underline{F}}_t(Y)$, integrating by parts and computing $\sigma(\sigma(\sigma(q/\varphi)')')'$,

$$\mathbb{E}[-\frac{1}{\varepsilon^3}(h(X_t)-h(M_t))^3 + \frac{3}{\varepsilon^2}\sigma h'(X_t)(h(X_t)-h(M_t)) \tag{6.29}$$

$$-\frac{1}{\varepsilon}\sigma(\sigma h')'(X_t) \mid \underline{\underline{F}}_t(Y)] = O(\varepsilon^{-1/2} \vee \varepsilon^{\frac{1}{2}-4\gamma})$$

Moreover, with our choice of M_t, $h(M_t)=\widehat{h}_t$, and from (6.20),

$$\mathbb{E}[\sigma h'(X_t)(h(X_t)-\widehat{h}_t) \mid \underline{\underline{F}}_t(Y)] = \widehat{e}_t^{(\sigma h')\,h} \tag{6.30}$$

$$= \varepsilon\sigma(\sigma h')'(\widehat{X}_t) + O(\varepsilon^{3/2} \vee \varepsilon^{2-2\gamma})$$

Estimate (6.26) is then easily derived. To obtain the fourth moment, note that

$$\frac{1}{\varphi}(\sigma(\sigma(\sigma(h-h(M_t))\varphi)')')'(t,X_t) = O(\varepsilon^{-1/2} \vee \varepsilon^{-2\gamma}) \tag{6.31}$$

from which we deduce

$$\mathbb{E}[-\frac{1}{\varepsilon^3}(h(X_t)-h(M_t))^4+\frac{3}{\varepsilon^2}\sigma h'(X_t)(h(X_t)-h(M_t))^2 \mid \underline{\underline{F}}_t(Y)] \tag{6.32}$$

$$= O(\varepsilon^{-1/2} \vee \varepsilon^{-2\gamma})$$

where we have suppressed in the left side a term which was dominated by the right side. Estimate (6.27) then follows from (6.2). ■

Remark: Proposition 6.3 provides us with an asymptotic expression for the third moment as soon as $\gamma<3/8$ and this expression is of order ε^2; however when $\gamma \geq 3/8$, we have only proved that this moment is dominated by $\varepsilon^{7/2-4\gamma}$. When γ tends to 1/2, this quantity tends to $\varepsilon^{3/2}$ which is an estimate evident from proposition 2.2; therefore, we again notice that our method is less and less effective as $\gamma \to 1/2$.

Now, as in corollary 6.2, we can deduce other fourth conditional expectations and obtain the asymptotic expression

$$\hat{e}_t^{g_1 g_2 g_3 g_4} = 3\varepsilon^2 \frac{\sigma^2}{h'^2} g_1' g_2' g_3' g_4'(\hat{X}_t) + O(\varepsilon^{5/2} \vee \varepsilon^{3-2\gamma}) \tag{6.33}$$

for any functions $g_1, \ldots, g_4$ of $\underline{\underline{B}}(2)$. However, for the third moments, one has to be more cautious; we note that from Taylor's formula,

$$e_t^h = h'(\hat{X}_t) e_t^X + \frac{1}{2} h''(\hat{X}_t)(e_t^{XX} - \hat{e}_t^{XX}) + O(\varepsilon^{3/2}) \tag{6.34}$$

so

$$\hat{e}_t^{hhh} = h'^3(\hat{X}_t)\hat{e}_t^{XXX} + \frac{3}{2} h'^2 h''(\hat{X}_t)(\hat{e}_t^{XXXX} - (\hat{e}_t^{XX})^2) + O(\varepsilon^{5/2}) \tag{6.35}$$

and therefore, combining corollary 6.2 and proposition 6.3,

$$\hat{e}_t^{XXX} = \varepsilon^2 \frac{2\sigma\sigma' h' - \sigma^2 h''}{h'^3}(\hat{X}_t) + O(\varepsilon^{5/2} \vee \varepsilon^{\frac{7}{2}-4\gamma}) \tag{6.36}$$

Similarly, one obtains

Proposition 6.4 *Assume (A3) and let g_1, g_2 and g_3 be three functions of $\underline{\underline{B}}(3)$. Then*

$$\widehat{e}_t^{g_1 g_2 g_3} = \varepsilon^2 \left[g_1' g_2' g_3' \frac{2\sigma\sigma' h' - \sigma^2 h''}{h'^3} + (g_1' g_2' g_3')' \frac{\sigma^2}{h'^2} \right] (\widehat{X}_t) \tag{6.37}$$

$$+ O(\varepsilon^{5/2} \vee \varepsilon^{\frac{7}{2} - 4\gamma})$$

In the following sections, we will need the two particular cases:

$$\widehat{e}_t^{hXX} = 2\varepsilon^2 \frac{\sigma\sigma'}{h'} (\widehat{X}_t) + O(\varepsilon^{5/2} \vee \varepsilon^{\frac{7}{2} - 4\gamma}) \tag{6.38}$$

$$\widehat{e}_t^{hGG} = O(\varepsilon^{5/2} \vee \varepsilon^{\frac{7}{2} - 4\gamma}) \tag{6.39}$$

(remember that G is a primitive function of $1/\sigma$).

Remark: If b is linear and σ constant, then some terms disappear in the calculations, and (6.19) and (6.36) can be improved as

$$\widehat{e}_t^{XX} = \varepsilon \frac{\sigma}{h'} + O(\varepsilon^{2-2\gamma}) \quad \text{and} \quad \widehat{e}_t^{XXX} = O(\varepsilon^{3-2\gamma} \vee \varepsilon^{\frac{7}{2} - 4\gamma}) \tag{6.40}$$

7. A two-dimensional filter

We are going to use the previous section to improve the approximation of $\widehat{e}_t^{XX}$ provided by corollary 6.2 and the approximation of $\widehat{X}_t$ provided by theorem 5.1. To this end, we have to find a good approximate equation for the vector $(\widehat{X}_t, \widehat{e}_t^{XX})$; the exact equation for $\widehat{X}_t$ is (6.22) and for $\nu_t = \widehat{e}_t^{XX}/\varepsilon$, it is

$$d\nu_t = \frac{1}{\varepsilon^{1-2\gamma}} \left[\left(-\left(\frac{\widehat{e}_t^{hX}}{\widehat{e}_t^{XX}} \right)^2 \nu_t^2 + 2\varepsilon^{1-2\gamma} \frac{\widehat{e}_t^{bX}}{\widehat{e}_t^{XX}} \nu_t + \widehat{\sigma^2}(X_t) \right) dt + \frac{\widehat{e}_t^{hXX}}{\varepsilon^2} (dY_t - \widehat{h}_t \, dt) \right] . \tag{7.1}$$

From (6.38), a natural approximation of $(\widehat{X}_t, \nu_t)$ would be the solution (M_t, Q_t) of

$$dM_t = b(M_t) dt + \frac{Q_t}{\varepsilon^{1-2\gamma}} h'(M_t) (dY_t - h(M_t) dt - \frac{\varepsilon}{2} Q_t h''(M_t) dt) \tag{7.2}$$

$$dQ_t = \frac{1}{\varepsilon^{1-2\gamma}}(-h'^2(M_t)Q_t^2+2\varepsilon^{1-2\gamma}b'(M_t)Q_t+\sigma^2(M_t))dt \tag{7.3}$$

$$+\frac{2}{\varepsilon^{1-2\gamma}}\frac{\sigma\sigma'}{h'}(M_t)(dY_t-h(M_t)dt)$$

However, unless σ is constant, the equation for the approximate variance involves a dY_t term so Q_t may become negative and the filter may diverge; if we neglect this term, the resulting filter will not be as good as one could expect. To overcome this difficulty, we can for instance multiply this term by $\varphi(Q_t)$ where φ is a smooth function which is equal to 1 on the interval $[\inf_x \sigma/(2h')(x),+\infty[$ and to 0 on a neighbourhood of zero; then Q_t will remain uniformly positive; moreover, since Q_t will be close to $\sigma/h'(M_t)$, the probability of the event $\varphi(Q_t)\neq 1$ will be very small, so the dY_t term will be well approximated. Here, we are going to use an alternative method: we will write an approximate equation for the vector $(\hat{X}_t,\hat{e}_t^{GC})$; in this case indeed, from (6.39), the dY_t coefficient of the second component can be neglected.

Theorem 7.1 *Assume (A3). Let m_0 and $r_0>0$ be real numbers and let (M_t,R_t) be the solution of the system*

$$dM_t = b(M_t)dt + \frac{R_t}{\varepsilon^{1-2\gamma}}\sigma^2 h'(M_t)(dY_t-h(M_t)dt-\frac{\varepsilon}{2}\frac{\sigma h''}{h'}(M_t)dt) \tag{7.4}$$

$$\frac{dR_t}{dt} = -\frac{1}{\varepsilon^{1-2\gamma}}\sigma^2 h'^2(M_t)R_t^2 + 2(\frac{b}{\sigma})'\sigma(M_t)R_t + \frac{1}{\varepsilon^{1-2\gamma}} \tag{7.5}$$

with the initial condition $(M_0,R_0)=(m_0,r_0)$. Then

$$\hat{e}_t^{XX} = \varepsilon\sigma^2(M_t)R_t + O(\varepsilon^2\vee\varepsilon^{3-4\gamma}) \tag{7.6}$$

$$\hat{X}_t = M_t + O(\varepsilon^{3/2}\vee\varepsilon^{\frac{5}{2}-4\gamma}) \tag{7.7}$$

Remark that the estimate given by (7.7) is always better than the one obtained in section 5 for a one-dimensional filter; it is dominated by $\varepsilon^{3/2}$ if $\gamma\leq 1/4$ and by ε if $\gamma\leq 3/8$.

Proof

First, note that equation (7.4) can be written in the form (2.18) with

$$U_t = -\frac{\varepsilon^{2\gamma}}{2}\sigma^3 h''(M_t)R_t \quad \text{and} \quad V_t = \sigma^2 h'(M_t)R_t \tag{7.8}$$

On the other hand, one can easily derive from (7.5) that the process R_t evolves between two strictly positive constants, so the assumptions of proposition 2.2 are satisfied. We also derive from (7.4), (7.5) and Itô's formula that

$$d\left[R_t - \frac{1}{\sigma h'}(M_t)\right] = \frac{1}{\varepsilon^{1-2\gamma}}\left[-A_t(R_t - \frac{1}{\sigma h'}(M_t))dt + B_t dt + C_t(dY_t - h(M_t)dt)\right] \tag{7.9}$$

with

$$A_t = \sigma^2 h'^2(M_t)R_t + \sigma h'(M_t) \tag{7.10}$$

$$B_t = 2\varepsilon^{1-2\gamma}(\frac{b}{\sigma})'\sigma(M_t)R_t - \varepsilon^{1-2\gamma}(\frac{1}{\sigma h'})'(M_t)(b(M_t)+U_t) - \frac{\varepsilon}{2}(\frac{1}{\sigma h'})''(M_t)V_t^2 \tag{7.11}$$

$$= O(\varepsilon^{1-2\gamma} \vee \varepsilon)$$

$$C_t = -(\frac{1}{\sigma h'})'(M_t)V_t = O(1) \tag{7.12}$$

We deduce from these equations and corollary 2.4 that

$$R_t = \frac{1}{\sigma h'}(M_t) + O(\varepsilon^{1/2} \vee \varepsilon^{1-2\gamma}) \tag{7.13}$$

Now this estimate implies that we can apply theorem 5.2 and obtain

$$\widehat{X}_t - M_t = O(\varepsilon \vee \varepsilon^{\frac{3}{2}-2\gamma}) \tag{7.14}$$

Let us write the exact equation for $\widehat{e}_t^{GG}$:

$$d\widehat{e}_t^{GG} = -\frac{(\widehat{e}_t^{Gh})^2}{\varepsilon^{2-2\gamma}}dt + 2\widehat{e}_t^{G\Phi}dt + \varepsilon^{2\gamma}dt + \frac{\widehat{e}_t^{GGh}}{\varepsilon^{2-2\gamma}}(dY_t - \widehat{h}_t dt) \tag{7.15}$$

with the notation $\Phi = \underline{\underline{L}}G = (b/\sigma) - \varepsilon^{2\gamma}\sigma'$. If we consider the process

$$\mu_t = \widehat{e}_t^{GG}/\varepsilon \tag{7.16}$$

then we can write (7.15) in the form

$$d\mu_t = \frac{1}{\varepsilon^{1-2\gamma}}(-a_t\mu_t^2 + b_t\mu_t + 1)dt + \frac{c_t}{\varepsilon^{1-2\gamma}}(dY_t - \widehat{h}_t dt) \tag{7.17}$$

with $a_t = (\widehat{e}_t^{Gh}/\widehat{e}_t^{GG})^2$, $b_t = 2\varepsilon^{1-2\gamma}\widehat{e}_t^{G\Phi}/\widehat{e}_t^{GG}$ and $c_t = \widehat{e}_t^{GGh}/\varepsilon^2$. On the other hand, by developing the function G,

$$\hat{e}_t^{GG} = \frac{\hat{e}_t^{GX}}{\sigma(\hat{X}_t)} - \frac{1}{2}\frac{\sigma'}{\sigma^2}(\hat{X}_t)\hat{e}_t^{GXX} + O(\varepsilon^2) \tag{7.18}$$

$$= \frac{\hat{e}_t^{GX}}{\sigma(\hat{X}_t)} + O(\varepsilon^2 \vee \varepsilon^{\frac{7}{2}-4\gamma})$$

from the results of previous section on the third moments. Similarly,

$$\hat{e}_t^{Gh} = \hat{e}_t^{GX} h'(\hat{X}_t) + O(\varepsilon^2 \vee \varepsilon^{\frac{7}{2}-4\gamma}) \tag{7.19}$$

and from (7.18), (7.19) and (7.14), we deduce

$$\mu_t^2 a_t = \mu_t^2 \sigma^2 h'^2(\hat{X}_t) + O(\varepsilon \vee \varepsilon^{\frac{5}{2}-4\gamma}) \tag{7.20}$$

$$= \mu_t^2 \sigma^2 h'^2(M_t) + O(\varepsilon \vee \varepsilon^{\frac{3}{2}-2\gamma})$$

With the same method,

$$\mu_t b_t = 2\varepsilon^{1-2\gamma}\mu_t\, \sigma\Phi'(\hat{X}_t) + O(\varepsilon^{2-2\gamma} \vee \varepsilon^{\frac{7}{2}-6\gamma}) \tag{7.21}$$

$$= 2\varepsilon^{1-2\gamma}\mu_t\, \sigma(\frac{b}{\sigma})'(\hat{X}_t) + O(\varepsilon \vee \varepsilon^{\frac{7}{2}-6\gamma})$$

$$= 2\varepsilon^{1-2\gamma}\mu_t\, \sigma(\frac{b}{\sigma})'(M_t) + O(\varepsilon \vee \varepsilon^{\frac{5}{2}-4\gamma})$$

and finally, from (6.39),

$$c_t = O(\varepsilon^{1/2} \vee \varepsilon^{\frac{3}{2}-4\gamma}) \tag{7.22}$$

Now, if we write the equation for $R-\mu$ in the form

$$d(R_t-\mu_t) = -\frac{1}{\varepsilon^{1-2\gamma}}\left[\sigma^2 h'^2(M_t)(R_t+\mu_t)+2\varepsilon^{1-2\gamma}(\frac{b}{\sigma})'\sigma(M_t)\right](R_t-\mu_t)dt \tag{7.23}$$

$$+ \frac{\mu_t^2}{\varepsilon^{1-2\gamma}}(a_t-\sigma^2 h'^2(M_t))dt - \frac{\mu_t}{\varepsilon^{1-2\gamma}}(b_t-2\varepsilon^{1-2\gamma}\sigma(\frac{b}{\sigma})'(M_t))dt$$

$$- \frac{c_t}{\varepsilon^{1-2\gamma}}(dY_t-\hat{h}_t dt)$$

then the coefficient of $(R-\mu)dt$ is uniformly positive if ε is sufficiently small, so we can apply corollary 2.4 and with the estimates (7.20)-(7.22), we obtain

$$R_t - \mu_t = O(\varepsilon \vee \varepsilon^{2-4\gamma}) \tag{7.24}$$

On the other hand,

$$\mu_t = \frac{\hat{e}_t^{GG}}{\varepsilon} = \frac{\hat{e}_t^{XX}}{\varepsilon\sigma^2(\hat{X}_t)} + O(\varepsilon \vee \varepsilon^{\frac{5}{2}-4\gamma}) = \frac{\hat{e}_t^{XX}}{\varepsilon\sigma^2(M_t)} + O(\varepsilon \vee \varepsilon^{\frac{3}{2}-2\gamma}) \tag{7.25}$$

so (7.6) follows from (7.24) and (7.25). Then we write the equation for $\hat{X}_t - M_t$ as

$$d(\hat{X}_t - M_t) = -\frac{R_t}{\varepsilon^{1-2\gamma}}\sigma^2 h'(M_t)(h(\hat{X}_t) - h(M_t))dt \tag{7.26}$$

$$- \frac{R_t}{\varepsilon^{1-2\gamma}}\sigma^2 h'(M_t)(\hat{h}_t - h(\hat{X}_t) - \frac{\varepsilon}{2}\frac{\sigma h''}{h'}(M_t))dt$$

$$+ (\hat{b}_t - b(M_t))dt + \frac{1}{\varepsilon^{1-2\gamma}}(\frac{\hat{e}_t^{hX}}{\varepsilon} - R_t\sigma^2 h'(M_t))(dY_t - \hat{h}_t dt)$$

Remark that if we note

$$\zeta_t = R_t\sigma^2 h'(M_t)(h(\hat{X}_t) - h(M_t)) \tag{7.27}$$

and $Z_t = \hat{X}_t - M_t$, then, since h' is uniformly positive, the assumption (2.9) of lemma 2.1 is satisfied. Let us estimate the other terms. From corollary 6.2,

$$\hat{h}_t - h(\hat{X}_t) - \frac{\varepsilon}{2}\frac{\sigma h''}{h'}(M_t) = O(\varepsilon^{3/2} \vee \varepsilon^{2-2\gamma}) \tag{7.28}$$

We also have

$$\varepsilon^{1-2\gamma}(\hat{b}_t - b(M_t)) = O(\varepsilon^{2-2\gamma} \vee \varepsilon^{\frac{5}{2}-4\gamma}) \tag{7.29}$$

and from (7.6),

$$\frac{\hat{e}_t^{hX}}{\varepsilon} - R_t\sigma^2 h'(M_t) = O(\varepsilon \vee \varepsilon^{2-4\gamma}) \tag{7.30}$$

Now (7.7) follows from corollary 2.4. ■

Remarks:

1. If b is linear and σ constant, then by means of (6.40), evaluations (7.6) and (7.7) can be improved as

$$\hat{e}_t^{XX} = \varepsilon\sigma^2 R_t + O(\varepsilon^{\frac{5}{2}-2\gamma} \vee \varepsilon^{3-4\gamma}) \tag{7.31}$$

$$\widehat{X}_t = M_t + O(\varepsilon^{2-2\gamma} \vee \varepsilon^{\frac{5}{2}-4\gamma}) \tag{7.32}$$

2. One can go on with this technique and derive more accurate filters by increasing their dimension. However (see the beginning of this section), one has to be cautious with the sign of the approximate conditional variance and modify the filter so that it remains positive.

8. Comparison with classical filters

Now, we are going to compare the filter obtained in previous section with some classical two-dimensional filters given in [5] and [2]. We will try to make clear the reasons for the differences of quality between these filters. However, we have to note that, since we have only upper bounds for the approximation errors, a comparison of these bounds cannot prove that a filter is necessarily better than another. All the filters are given by an approximate equation for $(\widehat{X}_t, \widehat{e}_t^{XX})$; they will be studied with the same method we have previously used and we will only give a sketch of the proofs for the first one.

8.1. Extended Kalman filter

The extended Kalman filter is the approximation $(M_t, \varepsilon Q_t)$ of $(\widehat{X}_t, \widehat{e}_t^{XX})$ which is given by the equations

$$dM_t = b(M_t)dt + \frac{Q_t}{\varepsilon^{1-2\gamma}} h'(M_t)(dY_t - h(M_t)dt) \tag{8.1}$$

$$\frac{dQ_t}{dt} = \frac{1}{\varepsilon^{1-2\gamma}} (-h'^2(M_t)Q_t^2 + 2\varepsilon^{1-2\gamma} b'(M_t)Q_t + \sigma^2(M_t)) \tag{8.2}$$

Like previously, we use some initial conditions $(M_0, Q_0) = (m_0, q_0)$, $q_0 > 0$ which are not very important for our estimates. By proceeding as in previous section, we prove that

$$Q_t = \frac{\sigma}{h'}(M_t) + O(\varepsilon^{1/2} \vee \varepsilon^{1-2\gamma}) \tag{8.3}$$

and that the estimate of theorem 5.1 holds. The exact equation for $\nu_t = \widehat{e}_t^{XX}/\varepsilon$ is (7.1) and we study $Q - \nu$ with the method we have used for $R - \mu$ in previous section; we have to compare the coefficients of equations (8.2) and (7.1). The differences between the coefficients of the ordinary Ricatti part of the equations are dominated by $\varepsilon \vee \varepsilon^{\frac{3}{2}-2\gamma}$, but, in the stochastic part, $\widehat{e}_t^{hXX}$ is neglected; for this term, we use the estimate (6.38), and unless σ is constant, we only have

$$\frac{\widehat{e}_t^{hXX}}{\varepsilon^2} = O(1 \vee \varepsilon^{\frac{3}{2}-4\gamma}) \tag{8.4}$$

This implies that

$$Q_t - \nu_t = O(\varepsilon^{1/2} \vee \varepsilon^{2-4\gamma}) \tag{8.5}$$

so

$$\frac{\widehat{e}_t^{hX}}{\varepsilon} = Q_t h'(M_t) + O(\varepsilon^{1/2} \vee \varepsilon^{2-4\gamma}) \tag{8.6}$$

Then we write the equation for $\widehat{X} - M$ as

$$\begin{aligned} d(\widehat{X}_t - M_t) = &-\frac{Q_t h'(M_t)}{\varepsilon^{1-2\gamma}}(h(\widehat{X}_t) - h(M_t))dt - \frac{Q_t h'(M_t)}{\varepsilon^{1-2\gamma}}(\widehat{h}_t - h(\widehat{X}_t))dt \\ &+ (\widehat{b}_t - b(M_t))dt + \frac{1}{\varepsilon^{1-2\gamma}}\left(\frac{\widehat{e}_t^{hX}}{\varepsilon} - Q_t h'(M_t)\right)(dY_t - \widehat{h}_t dt) \end{aligned} \tag{8.7}$$

and deduce

$$\widehat{X}_t = M_t + O(\varepsilon \vee \varepsilon^{\frac{5}{2}-4\gamma}) \tag{8.8}$$

Comparing (7.7) and (8.8), we note that (8.8) is worse than (7.7) if $\gamma < 3/8$ but they are identical if $\gamma \geq 3/8$; in particular, the extended Kalman filter leads to an error of order ε provided that $\gamma \leq 3/8$. We also note that the loss of efficiency for small values of γ has two origins; first, in the equation of $\widehat{e}_t^{XX}$, the coefficient $\widehat{e}_t^{hXX}$ of $dY_t - \widehat{h}_t dt$ has been neglected, and second, in the equation of $\widehat{X}_t$, the term $\widehat{h}_t$ has been replaced by $h(M_t)$ whereas the filter of section 7 involved also the second derivative of h. The first defect disappears if σ is constant, the second if h is linear, so the estimate is not improved if only one of these conditions is satisfied. If both of them are satisfied, then the extended Kalman filter coincides with the filter of previous section.

8.2. Statistical linearization

For a function g and two real numbers m and $Q>0$, define

$$\Psi(g,m,Q) = \frac{1}{\sqrt{2\pi Q}}\int_{-\infty}^{+\infty} g(x)\exp(-\frac{(x-m)^2}{2Q})dx \tag{8.9}$$

if the integral is defined. Roughly speaking, the statistical linearization consists of replacing, in the state and observation equations, the processes $b(X_t)$, $\sigma(X_t)$ and $h(X_t)$ respectively by $\Psi(b,M_t,\varepsilon Q_t)+\Psi(b',M_t,\varepsilon Q_t)(X_t-M_t)$, $\Psi(\sigma,M_t,\varepsilon Q_t)$ and $\Psi(h,M_t,\varepsilon Q_t)+\Psi(h',M_t,\varepsilon Q_t)(X_t-M_t)$, where $(M_t,\varepsilon Q_t)$ is an approximation of $(\hat{X}_t,\hat{e}_t^{XX})$. The equations are

$$dM_t = \Psi(b,M_t,\varepsilon Q_t)dt + \frac{Q_t}{\varepsilon^{1-2\gamma}}\Psi(h',M_t,\varepsilon Q_t)(dY_t-\Psi(h,M_t,\varepsilon Q_t)dt) \tag{8.10}$$

$$\frac{dQ_t}{dt} = \frac{1}{\varepsilon^{1-2\gamma}}(-Q_t^2\Psi^2(h',M_t,\varepsilon Q_t)+2Q_t\varepsilon^{1-2\gamma}\Psi(b',M_t,\varepsilon Q_t)+\Psi^2(\sigma,M_t,\varepsilon Q_t)) \tag{8.11}$$

One can prove that this filter satisfies the same estimate (8.5)-(8.8) than the extended Kalman filter. More precisely, by replacing, in the equation for M, $h(M_t)$ by $\Psi(h,M_t,\varepsilon Q_t)$, we have suppressed the second defect, but not the first one; however, if σ is constant, then $\hat{e}_t^{hXX}$ can be neglected so the statistical linearization works like the filter of section 7.

8.3. The truncated, Gaussian and modified second-order filters

The so-called truncated second-order filter is obtained by neglecting the third and fourth conditional moments $\hat{e}_t^{XXX}$ and $\hat{e}_t^{XXXX}$; the Gaussian filter is obtained by neglecting $\hat{e}_t^{XXX}$ and assuming $\hat{e}_t^{XXXX}\approx 3(\hat{e}_t^{XX})^2$. For these two filters, the approximate variance equation has a stochastic part, so it seems difficult to study the the sign of the approximate variance and, if it happens to become negative, the filter may diverge. Moreover, it is noticed in [3] that the truncated filter involves an illogical approximation. So one generally rather considers the so-called modified second-order filter whose equations are

$$dM_t = (b(M_t)+\frac{\varepsilon}{2}Q_t b''(M_t))dt + \frac{Q_t}{\varepsilon^{1-2\gamma}}h'(M_t)(dY_t-h(M_t)dt-\frac{\varepsilon}{2}Q_t h''(M_t)dt) \tag{8.12}$$

$$\frac{dQ_t}{dt} = \frac{1}{\varepsilon^{1-2\gamma}}(-Q_t^2h'^2(M_t)+2\varepsilon^{1-2\gamma}b'(M_t)Q_t+\sigma^2(M_t)+\varepsilon Q_t(\sigma\sigma''+\sigma'^2)(M_t)) \tag{8.13}$$

We still use the same technique to study it and it turns out that it behaves like the statistical linearization: it satisfies (8.8) in the general case, but (7.7) if σ is constant and (7.32) if σ is constant and h is linear.

9. Summary of results

The three tables give the estimate for $\hat{X}_t - M_t$ for the considered filters, according to the value of γ and the properties of the coefficients.

TABLE I $\gamma \leq 1/4$

	General case	σ constant	σ constant h linear
Filter (5.1)	$O(\varepsilon)$	$O(\varepsilon)$	$O(\varepsilon^{\frac{3}{2}-2\gamma})$
Extended Kalman filter	$O(\varepsilon)$	$O(\varepsilon)$	$O(\varepsilon^{2-2\gamma})$
Statistical linearization Modified 2nd order filter	$O(\varepsilon)$	$O(\varepsilon^{3/2})$	$O(\varepsilon^{2-2\gamma})$
Filter (7.4)-(7.5)	$O(\varepsilon^{3/2})$	$O(\varepsilon^{3/2})$	$O(\varepsilon^{2-2\gamma})$

TABLE II $1/4 \leq \gamma \leq 3/8$

	General case	σ constant	σ constant h linear
Filter (5.1)	$O(\varepsilon^{\frac{3}{2}-2\gamma})$	$O(\varepsilon^{\frac{3}{2}-2\gamma})$	$O(\varepsilon^{\frac{3}{2}-2\gamma})$
Extended Kalman filter	$O(\varepsilon)$	$O(\varepsilon)$	$O(\varepsilon^{\frac{5}{2}-4\gamma})$
Statistical linearization Modified 2nd order filter	$O(\varepsilon)$	$O(\varepsilon^{\frac{5}{2}-4\gamma})$	$O(\varepsilon^{\frac{5}{2}-4\gamma})$
Filter (7.4)-(7.5)	$O(\varepsilon^{\frac{5}{2}-4\gamma})$	$O(\varepsilon^{\frac{5}{2}-4\gamma})$	$O(\varepsilon^{\frac{5}{2}-4\gamma})$

TABLE III	$3/8 \leq \gamma < 1/2$
	All the cases
Filter (5.1)	$O(\varepsilon^{\frac{3}{2}-2\gamma})$
Extended Kalman filter Statistical linearization Modified 2nd order filter Filter (7.4)-(7.5)	$O(\varepsilon^{\frac{5}{2}-4\gamma})$

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